Strong laws under trimming - a comparison between iid random variables and dynamical systems

Tanja Schindler

The Australian National University
soon Scuola Normale Superiore di Pisa

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1. The finite expectation case
2. The example $\mathbb{P}(X_1 > x) = L(x)/x$
3. The example $\mathbb{P}(X_1 > x) = L(x)/x^{\alpha}$, $\alpha \in (0,1)$
   - The i.i.d. case
   - The dynamical systems case
   - Mean convergence

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Theorem 1.2 ([Feller, 1946, Chow and Robbins, 1961])

Let (X_n) be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) < \infty$. Given any sequence of constants $(d_n)_{n\in\mathbb{N}}$ with $d_n > 0$ for all n, then

$$\limsup_{n\to\infty} \frac{S_n}{d_n} = \infty \text{ a.s. or } \liminf_{n\to\infty} \frac{S_n}{d_n} = 0 \text{ a.s.}$$

In the ergodic case we have analog statements:

Theorem 1.3 (Ergodic Theorem [Birkhoff, 1931])

Let $(\Omega, \mathcal{B}, \mu, T)$ be an ergodic, probability measure preserving dynamical system and let $f : \Omega \to \mathbb{R}$. If $(X_n) = \left(f \circ T^{n-1}\right)$ and $\mathbb{E}(X_1) < \infty$, then

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Theorem 1.4 ([Aaronson, 1977])

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Definition 2.1

The sum $S_n^{b_n}$ is called

- *lightly trimmed sum* if $b_n = r \in \mathbb{N}$,
- *intermediately (moderately) trimmed sum* if $\lim_{n\to\infty} b_n = \infty$ and $b_n = o(n)$,
- *heavily trimmed sum* if $b_n \sim \kappa \cdot n$, $0 < \kappa < 1$.

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- If (X_n) are sufficiently fast ψ-mixing it follows from [Aaronson and Nakada, 2003, Application of Theorem 1.1] that

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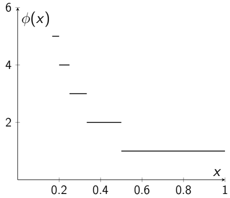
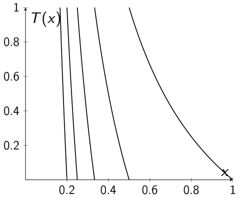
$$\phi\colon [0,1] \to \mathbb{R}_{>0}, \quad x \mapsto \lfloor 1/x \rfloor \qquad T\colon [0,1] \to [0,1], \quad x \mapsto 1/x \mod 1.$$





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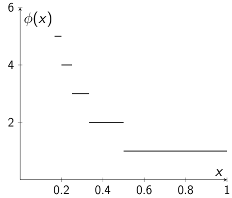

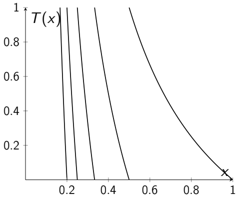

Then $a_n(x) = \phi \circ \tau^{n-1}(x)$.

Lemma 2.2 ([Diamond and Vaaler, 1986])

If $X_n := a_n$, $n \in \mathbb{N}$, then we have that

$$\lim_{n\to\infty} \frac{S_n^1}{n \log n} = \frac{1}{\log 2} \text{ a.s.}$$

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However, ψ-mixing is a strong condition on dynamical systems and not all interesting dynamical systems are ψ-mixing...

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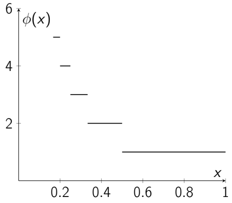
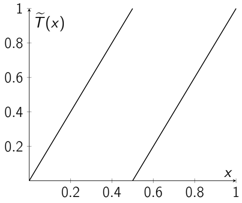
$$\phi\colon [0,1] \to \mathbb{R}_{>0}, \quad x \mapsto \lfloor 1/x \rfloor \qquad\qquad \widetilde{T}\colon [0,1] \to [0,1], \quad x \mapsto 2x \mod 1.$$





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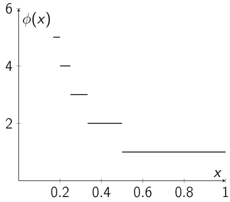
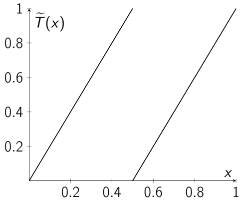


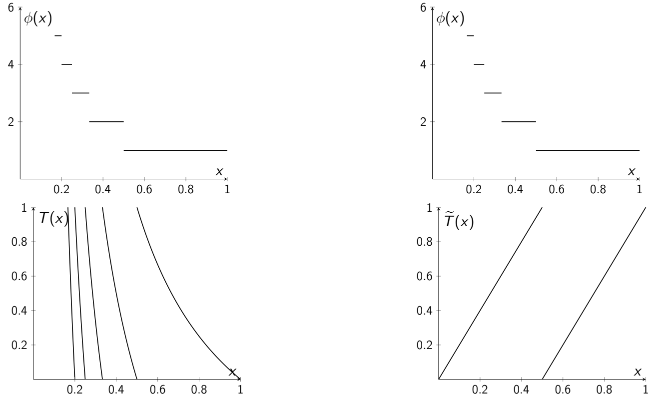


Theorem 2.3 ([Haynes, 2014, Theorem 4, generalized])

If $X_n = \phi \circ \widetilde{T}^{n-1}$, then for all positive valued sequences $(d_n)_{n\in\mathbb{N}}$ and $k \in \mathbb{N}$ we have (with respect to the Lebesgue measure λ) that either

$$\limsup_{n\to\infty} \frac{S_n^k}{d_n} = \infty \text{ a.s. or } \liminf_{n\to\infty} \frac{S_n^k}{d_n} = 0 \text{ a.s.}$$

Comparison: Continued fractions (left) / doubling map (right)



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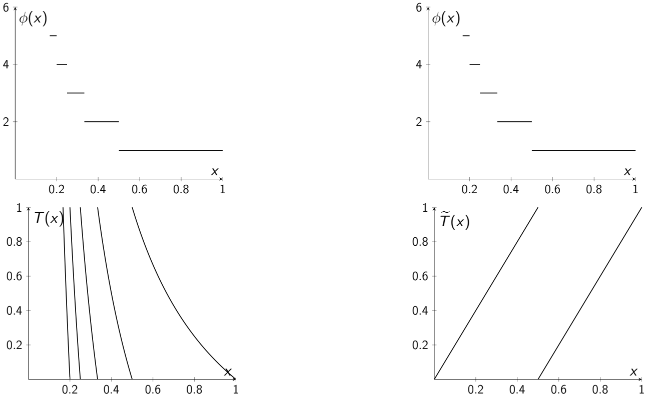

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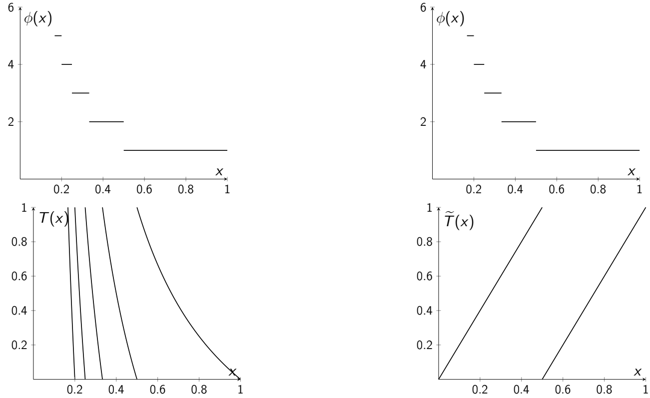

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If $\phi \circ \widetilde{T}^n > 1$ then $\phi \circ \widetilde{T}^{n+1} = \left\lfloor \frac{\phi \circ \widetilde{T}^{n+1}}{2} \right\rfloor$.

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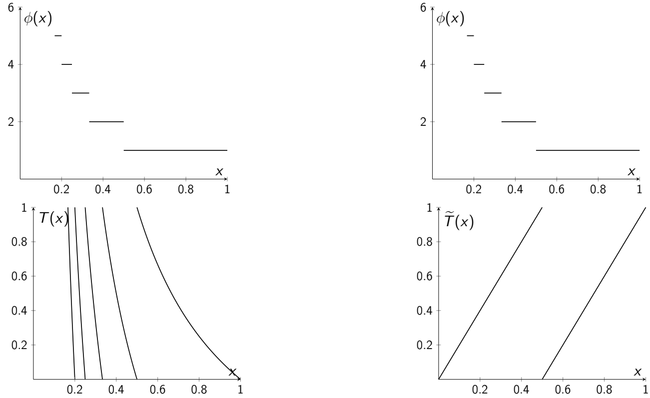

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So let's try intermediate trimming!

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Theorem 2.4 ([S., 2018, Theorem 1.1 & 1.2])

Let $(X_n) = (\phi \circ \widetilde{T}^{n-1})$ *and let* $\lim_{n\to\infty} b_n / \log^{1/4} n = 0$. *Iff there exists* $\psi \in \Psi$ *such that*

$$b_n := \left\lfloor \frac{\log \psi(\lfloor \log n \rfloor) - \log \log n}{\log 2} \right\rfloor,$$

then there exists a norming sequence (d_n) *such that*

$$\lim_{n\to\infty} \frac{S_n^{b_n}}{d_n} = 1 \text{ a.s.} \tag{1}$$

In case that (1) *holds we have that* $d_n = n \cdot \log n$.

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In particular we have that $b_n = \lfloor u \cdot \log \log \log n \rfloor$ for all $n \in \mathbb{N}$ is a possible trimming sequence if and only if $u > 1/\log 2$.

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It is work in progress to determine limit laws for more general settings than the doubling map and the observable ϕ.

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- If for example $F(x) = 1 - \exp\left(-\log^{3/2}(x)\right)$, then there exists (d_n) such that $\lim_{n\to\infty} \frac{S_n^2}{d_n} = 1$ a.s. while deleting only one digit does not work.

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- But there are also distribution functions $F(x) = 1 - L(x)/x$ such that there is no strong law of large numbers under light trimming, one example is given in [Aaronson and Nakada, 2003].
- To my knowledge for such functions not more is known neither in the i.i.d. nor in the dynamical systems setting.

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- Haeusler and Mason proved laws of the iterated logarithm for intermediately trimmed sums, see [Haeusler and Mason, 1987] and [Haeusler, 1993].

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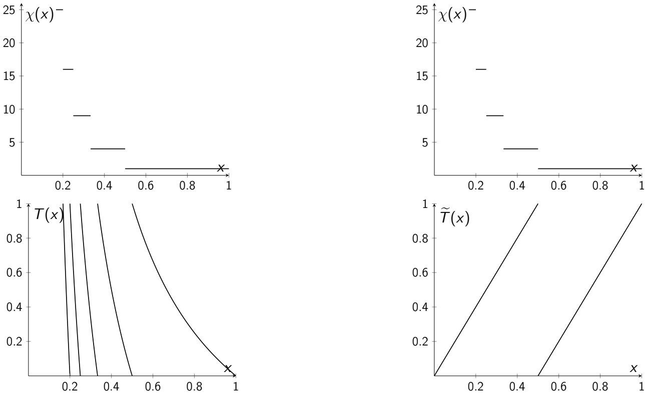
The norming sequence (d_n) can be given explicitly.

1. The finite expectation case
2. The example $\mathbb{P}(X_1 > x) = L(x)/x$
3. The example $\mathbb{P}(X_1 > x) = L(x)/x^{\alpha}$, $\alpha \in (0,1)$
   - The i.i.d. case
   - The dynamical systems case
   - Mean convergence

Consider two new systems: Define

$$\chi\colon [0,1] \to \mathbb{R}_{>0} \qquad T\colon [0,1] \to [0,1], \qquad \widetilde{T}\colon [0,1] \to [0,1],$$
$$x \mapsto \lfloor 1/x \rfloor^2 \qquad x \mapsto 1/x \mod 1 \qquad x \mapsto 2x \mod 1.$$



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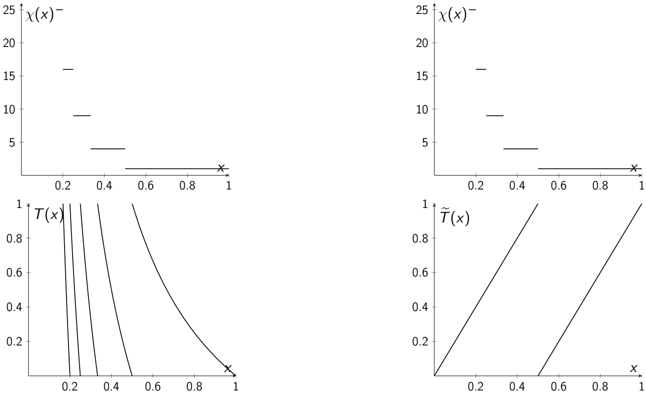

For $(X_n) = (\chi \circ T^{n-1})$ and for $(X_n) = (\chi \circ \widetilde{T}^{n-1})$ we obtain the same trimmed strong law:

Theorem 3.2 (Application of [Kesseböhmer and S., 2018, Theorem 1.7])

Let X_n be given as above. Further, let $(b_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers tending to infinity with $b_n = o(n)$. If

$$\lim_{n\to\infty} \frac{b_n}{\log\log n} = \infty, \tag{3}$$

then there exists a positive valued sequence $(d_n)_{n\in\mathbb{N}}$ such that

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In this case $d_n \sim \dfrac{n^2}{b_n}$.

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We note that in both cases $(X_n) = (\chi \circ T^{n-1})$ and $(X_n) = (\chi \circ \widetilde{T}^{n-1})$ the condition on the norming sequence (b_n) is the same as in the i.i.d. case.

- Indeed [Kesseböhmer and S., 2018] gives general conditions for dynamical systems fulfilling a spectral gap property on the transfer operator and observables with regularly varying tails with exponent strictly between 0 and 1 for a strong law under intermediate trimming to hold.

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- As an application we obtain strong laws under trimming for piecewise expanding interval maps.
- Another application of these results gives strong laws under trimming for subshifts of finite type, see [Kesseböhmer and S., 2019a].

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- The strong law of large nubmers or Birkhoff's ergodic theorem give for i.i.d. or ergodic and identically distributed, integrable random variables

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- In the generalized case for non-integrable random variables we obtain a strong law after trimming, i.e. there exists a (possibly constant) sequence of natural numbers (b_n) and a norming sequence (d_n) fulfilling

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- Can we say anything about the norming sequence (d_n)?
- In general: No!
  Even for i.i.d. random variables there are examples for which (5) holds but $\mathbb{E}\left(S_n^{b_n}\right) = \infty$, see [Kesseböhmer and S., 2019c, Remark 3].

However, remember the two systems from before:

$$\chi\colon\ [0,1] \to \mathbb{R}_{>0} \qquad T\colon\ [0,1] \to [0,1]\,, \qquad \widetilde{T}\colon\ [0,1] \to [0,1]\,,$$

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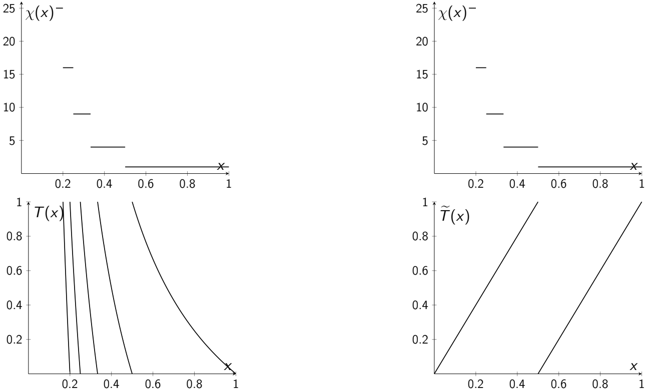

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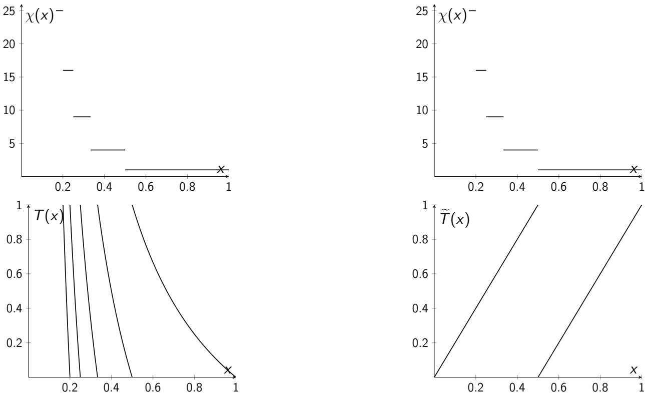

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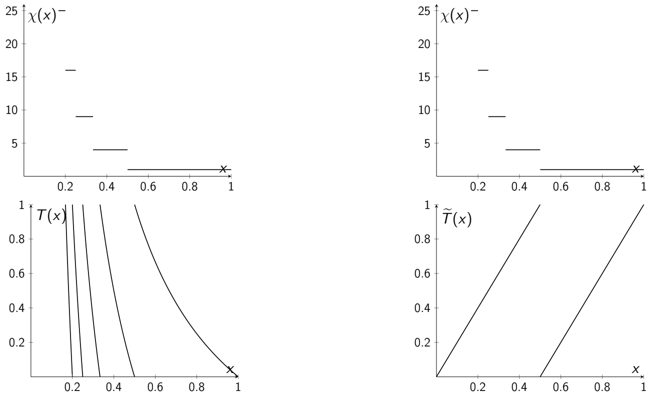

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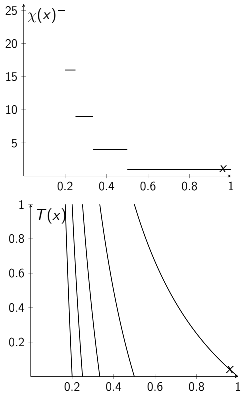

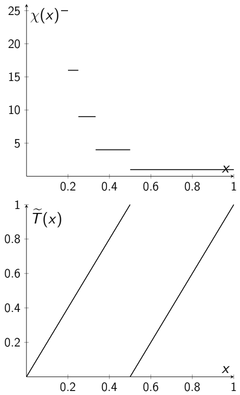

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For details see [Kesseböhmer and S., 2019b].

- [Kesseböhmer and S., 2019b] gives general conditions for mean convergence for the case $F(x) = 1 - L(x)/x^{\alpha}$, L slowly varying, $0 < \alpha < 1$.

- [Kesseböhmer and S., 2019b] gives general conditions for mean convergence for the case $F(x) = 1 - L(x)/x^\alpha$, L slowly varying, $0 < \alpha < 1$.
- If (X_n) are either independent or $X_n = f \circ T^{n-1}$ with f sufficiently regular and T a piecewise expanding interval map and (X_n) are exponentially fast ψ-mixing and additionally we have for (b_n) and (d_n) that

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  then we also have mean convergence.
- As seen before the ψ-mixing property is essential.

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