Homoclinic orbits in the Circular Restricted Four Body Problem

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Joint work with:

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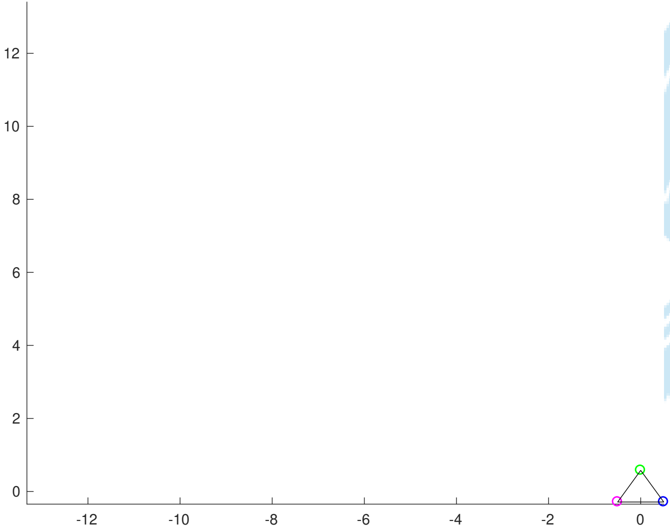
Goal:

Develop a numerical scheme to find and validate the existence of homoclinic orbits in the circular restricted four body problem at bifurcation points.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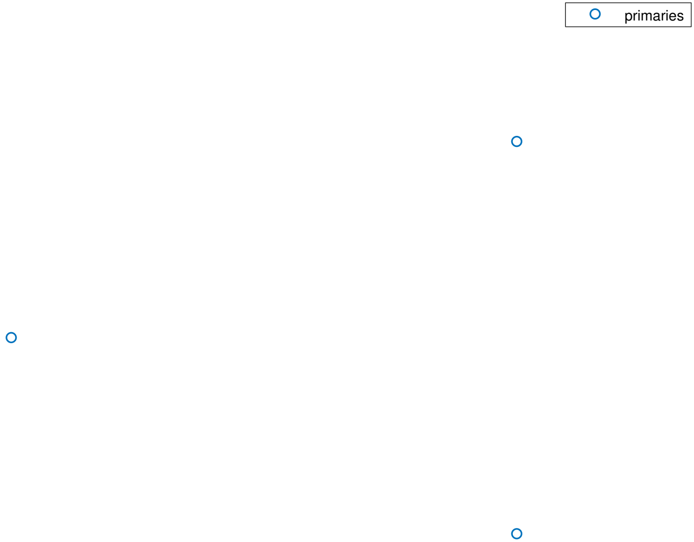
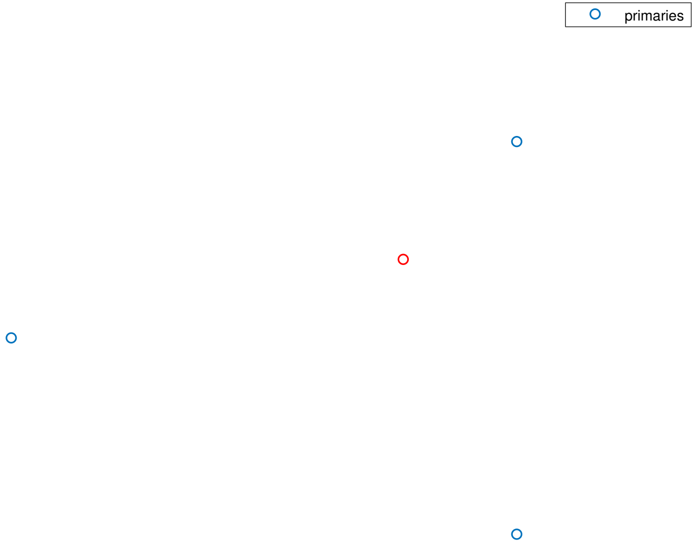

primaries


primaries

Dynamics in CRFBP

Laws of motion

$$\ddot{x} = 2\dot{y} + \partial_x \Omega(x, y)$$
$$\ddot{y} = -2\dot{x} + \partial_y \Omega(x, y)$$

with $\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=1}^{3} \frac{m_i}{\sqrt{(x-x_i)^2+(y-y_i)^2}}$.

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Position primary bodies

$$x_1 = -M \qquad y_1 = 0$$
$$x_2 = \frac{2m_2 + m_3 - 2M^2}{2M} \qquad y_2 = -\frac{\sqrt{3}m_3}{2M}$$
$$x_3 = \frac{m_2 + 2m_3 - 2M^2}{2M} \qquad y_3 = \frac{\sqrt{3}m_2}{2M}$$

with constant $M = \sqrt{m_2^2 + m_2 m_3 + m_3^3}$.

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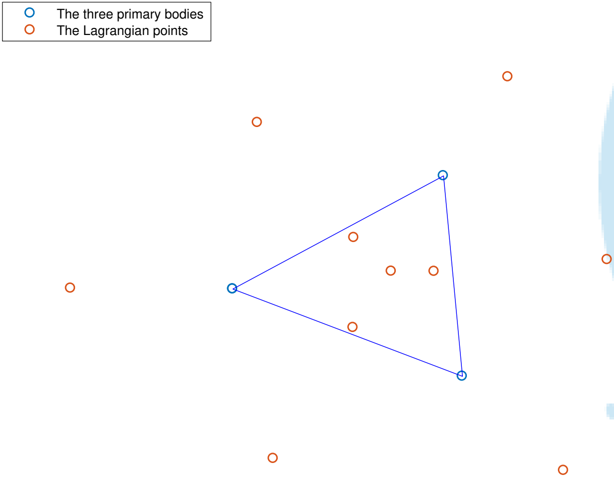
Assumptions on masses

$$\sum_{i=1}^{3} m_i = 1 \quad \text{and} \quad 0 < m_3 \leq m_2 \leq m_1 < 1.$$

Energy conservation

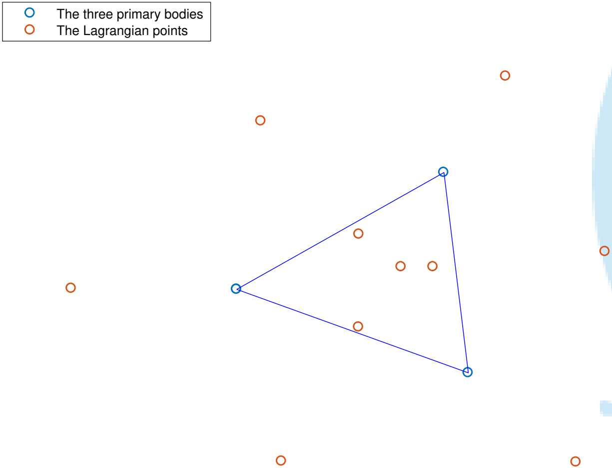
$$E := \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x, y).$$

Lagrangian points and bifurcations



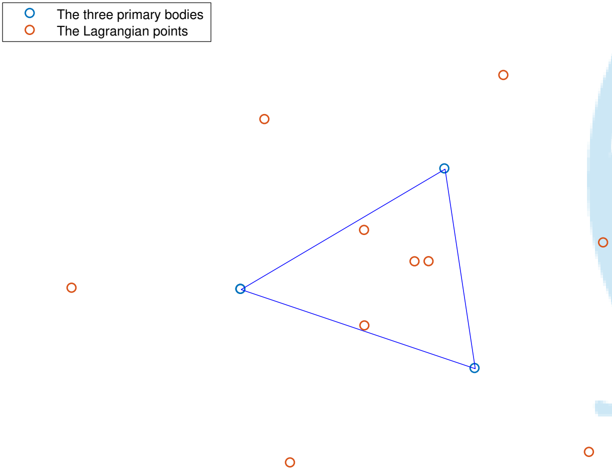
10 Lagrangian points for $m_1 = 0.38$ and $m_2 = 0.35$.

Lagrangian points and bifurcations



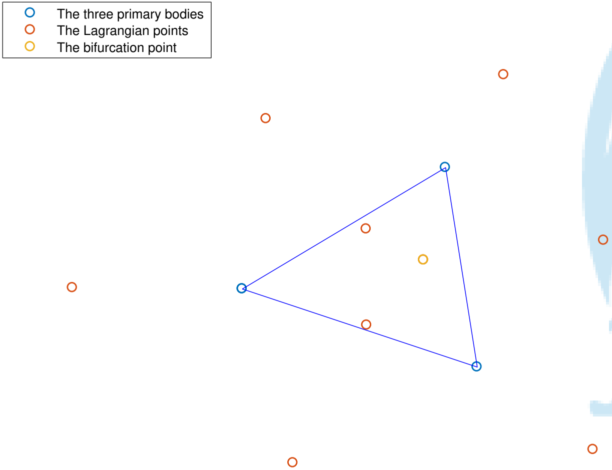
10 Lagrangian points for $m_1 = 0.40$ and $m_2 = 0.35$.

Lagrangian points and bifurcations



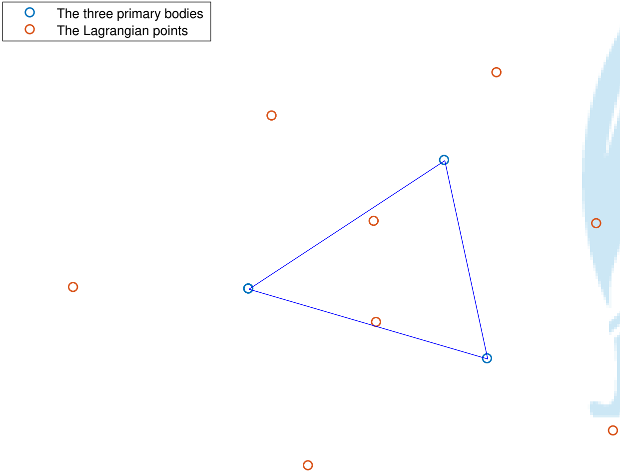
10 Lagrangian points for $m_1 = 0.42$ and $m_2 = 0.35$.

Lagrangian points and bifurcations



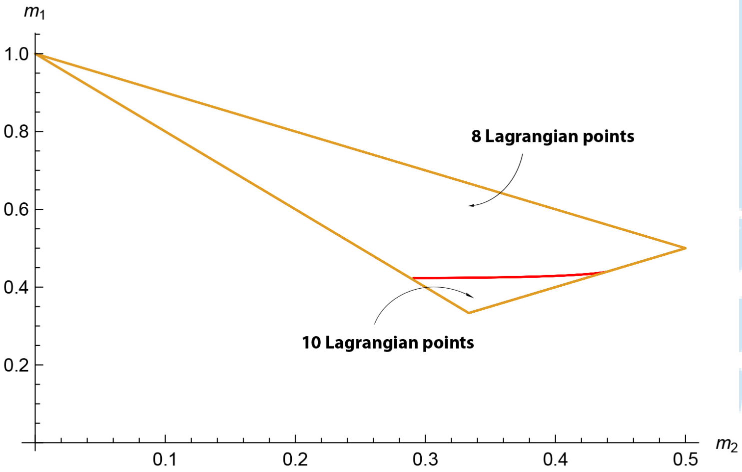
9 Lagrangian points for $m_1 \approx 0.4247$ and $m_2 = 0.35$.

Lagrangian points and bifurcations



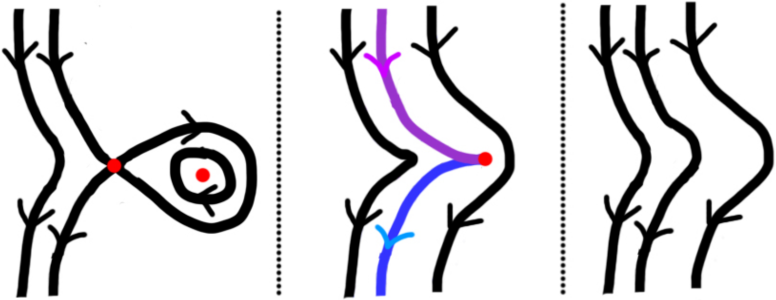
8 Lagrangian points for $m_1 = 0.46$ and $m_2 = 0.35$.

Lagrangian points and bifurcations



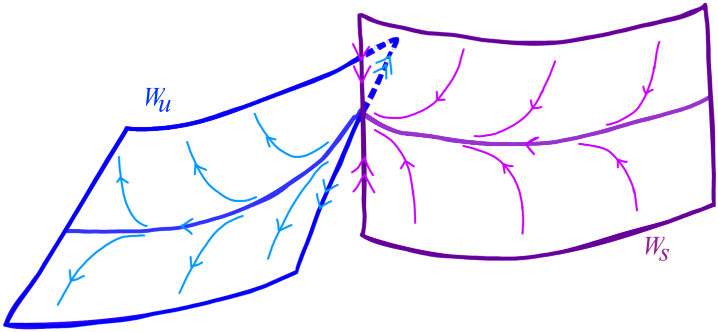
The bifurcation curve in the (m_1, m_2)-plane.

Hamiltonian saddle node bifurcation



Bifurcation diagram of a two dimensional Hamiltonian saddle node bifurcation.

Invariant Manifolds



The non-linear stable and unstable manifolds with their dynamics

Numerical approach to homoclinic orbits

1. Find a point on the bifurcation curve.
2. Compute the center, center-stable and center-unstable manifolds and conjugate dynamics (up to high order).
3. Compute the appropriate 2 dimensional manifolds W_u and W_s (up to high order).
4. Find a global connection between W_u and W_s.
5. Validate the solution.

Existence of invariant manifolds

Theorem (Parameterization method for center manifolds)

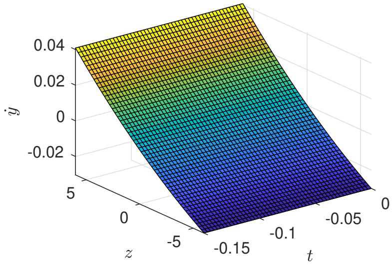
There exists a $K_c : \mathbb{R}^2 \to \mathbb{R}^4$ *and* $R_c : \mathbb{R}^2 \to \mathbb{R}^2$ *s.t.*

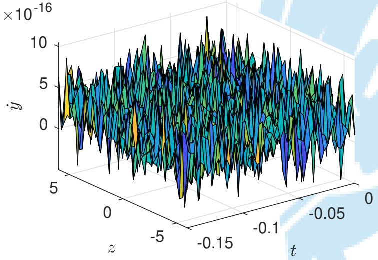
$$\begin{cases} F \circ K_c = DK_c \cdot R_c \text{ locally} \\ K_c \text{ is tangent to the center subspace} \\ R_c \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} + h.o.t. \end{cases}$$

Numerical computations

- Numerically compute 15^{th} order Taylor polynomials for the manifolds.
- Restrict the non-linear (un)stable manifold such that the error of the conjugacy equation is of the order 10^{-16}.
- Size of the non-linear (un)stable manifold is of the order 10^{-2}.
- Use an ODE-solver to find the homoclinic orbit.

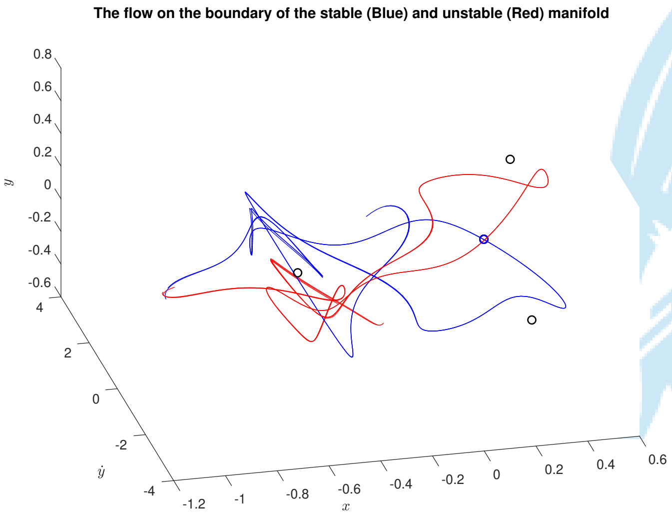
The invariant manifold





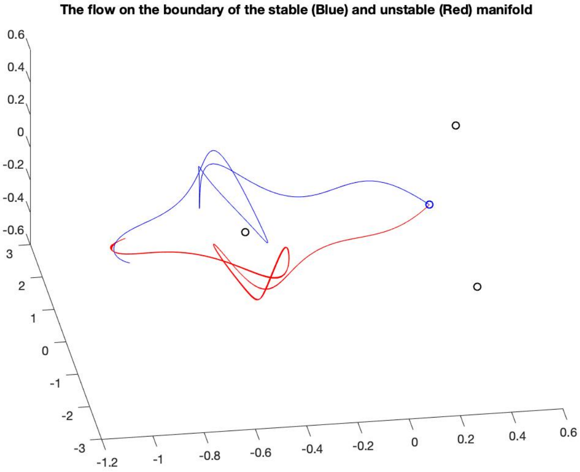
The $\dot{y}$-coordinate of the unstable manifold and its error.

The homoclinic orbit



The numerical homoclinic orbit.

The homoclinic orbit



The numerical homoclinic orbit.

Thanks for your attention