Strong laws under trimming - a comparison between iid random variables and dynamical systems

Tanja Schindler

The Australian National University
soon Scuola Normale Superiore di Pisa

7th Bremen summer school and symposium
dynamical systems - pure and applied,
August 2019
1. The finite expectation case

2. The example $\mathbb{P}(X_1 > x) = L(x)/x$

3. The example $\mathbb{P}(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$
   - The i.i.d. case
   - The dynamical systems case
   - Mean convergence
I will compare some limit theorems in the independent identically distributed (i.i.d.) setting with different dynamical systems.
I will compare some limit theorems in the independent identically distributed (i.i.d.) setting with different dynamical systems. Generally, let \((X_n)\) be a sequence of identically distributed random variables with distribution function \(F\) (i.e. \(F(x) = \mathbb{P}(X_1 \leq x)\)) and set \(S_n := \sum_{k=1}^{n} X_k\).
I will compare some limit theorems in the independent identically distributed (i.i.d.) setting with different dynamical systems. Generally, let $(X_n)$ be a sequence of identically distributed random variables with distribution function $F$ (i.e. $F(x) = \mathbb{P}(X_1 \leq x)$) and set $S_n := \sum_{k=1}^{n} X_k$.

**Theorem 1.1 (Strong law of large numbers)**

Let $(X_n)$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_1) < \infty$. Then

$$\lim_{n \to \infty} \frac{S_n}{n} = \mathbb{E}(X_1) \text{ a.s.}$$
I will compare some limit theorems in the independent identically distributed (i.i.d.) setting with different dynamical systems. Generally, let \((X_n)\) be a sequence of identically distributed random variables with distribution function \(F\) (i.e. \(F(x) = \mathbb{P}(X_1 \leq x)\)) and set \(S_n := \sum_{k=1}^{n} X_k\).

**Theorem 1.1 (Strong law of large numbers)**

*Let \((X_n)\) be a sequence of i.i.d. random variables with \(\mathbb{E}(X_1) < \infty\). Then*

\[
\lim_{n \to \infty} \frac{S_n}{n} = \mathbb{E}(X_1) \text{ a.s.}
\]

This holds no longer true if \(\mathbb{E}(X_1) = \infty\).
I will compare some limit theorems in the independent identically distributed (i.i.d.) setting with different dynamical systems. Generally, let \((X_n)\) be a sequence of identically distributed random variables with distribution function \(F\) (i.e. \(F(x) = \mathbb{P}(X_1 \leq x)\)) and set \(S_n := \sum_{k=1}^{n} X_k\).

**Theorem 1.1 (Strong law of large numbers)**

Let \((X_n)\) be a sequence of i.i.d. random variables with \(\mathbb{E}(X_1) < \infty\). Then

\[
\lim_{n \to \infty} \frac{S_n}{n} = \mathbb{E}(X_1) \text{ a.s.}
\]

This holds no longer true if \(\mathbb{E}(X_1) = \infty\).

**Theorem 1.2 ([Feller, 1946, Chow and Robbins, 1961])**

Let \((X_n)\) be a sequence of i.i.d. random variables with \(\mathbb{E}(X_1) < \infty\). Given any sequence of constants \((d_n)_{n \in \mathbb{N}}\) with \(d_n > 0\) for all \(n\), then

\[
\limsup_{n \to \infty} \frac{S_n}{d_n} = \infty \text{ a.s. or } \liminf_{n \to \infty} \frac{S_n}{d_n} = 0 \text{ a.s.}
\]
In the ergodic case we have analog statements:

**Theorem 1.3 (Ergodic Theorem [Birkhoff, 1931])**

Let $(\Omega, \mathcal{B}, \mu, T)$ be an ergodic, probability measure preserving dynamical system and let $f : \Omega \to \mathbb{R}$. If $(X_n) = (f \circ T^{n-1})$ and $\mathbb{E}(X_1) < \infty$, then

$$
\lim_{n \to \infty} \frac{S_n}{n} = \mathbb{E}(X_1) \quad \mu - \text{a.s.}
$$
In the ergodic case we have analog statements:

**Theorem 1.3 (Ergodic Theorem [Birkhoff, 1931])**

Let \((\Omega, \mathcal{B}, \mu, T)\) be an ergodic, probability measure preserving dynamical system and let \(f : \Omega \to \mathbb{R}\). If \((X_n) = (f \circ T^{n-1})\) and \(E(X_1) < \infty\), then

\[
\lim_{n \to \infty} \frac{S_n}{n} = E(X_1) \quad \mu - \text{a.s.}
\]

**Theorem 1.4 ([Aaronson, 1977])**

Let \((\Omega, \mathcal{B}, \mu, T)\) be an ergodic, probability measure preserving dynamical system and let \(f : \Omega \to \mathbb{R}\). Further, let \((X_n) = (f \circ T^{n-1})\) and \(E(X_1) = \infty\). Given any sequence of constants \((d_n)_{n \in \mathbb{N}}\) with \(d_n > 0\) for all \(n\), then

\[
\limsup_{n \to \infty} \frac{S_n}{d_n} = \infty \mu - \text{a.s.} \quad \text{or} \quad \liminf_{n \to \infty} \frac{S_n}{d_n} = 0 \quad \mu - \text{a.s.}
\]
1. The finite expectation case

2. The example $P(X_1 > x) = L(x)/x$

3. The example $P(X_1 > x) = L(x)/x^\alpha, \alpha \in (0, 1)$
   - The i.i.d. case
   - The dynamical systems case
   - Mean convergence
We might obtain a strong law of large numbers after trimming.
We might obtain a strong law of large numbers after trimming. Let $\pi \in S_n$ be pointwise defined such that

$$X_{\pi(1)} \geq \ldots \geq X_{\pi(n)}$$

is a rearrangement of $X_1, \ldots, X_n$. The example $P(X_1 > x) = L(x)/x$. 

We might obtain a strong law of large numbers after trimming.
Let \( \pi \in S_n \) be pointwise defined such that
\[
X_{\pi(1)} \geq \ldots \geq X_{\pi(n)} \text{ is a rearrangement of } X_1, \ldots, X_n.
\]
In some literature the order statistics is also denoted by \( X_{n,1} \geq \ldots \geq X_{n,n} \).
We might obtain a strong law of large numbers after trimming. Let \( \pi \in S_n \) be pointwise defined such that

\[
X_{\pi(1)} \geq \ldots \geq X_{\pi(n)}
\]

is a rearrangement of \( X_1, \ldots, X_n \).

In some literature the order statistics is also denoted by \( X_{n,1} \geq \ldots \geq X_{n,n} \).

For the following let \((b_n)\) always be a sequence of numbers in \( \mathbb{N}_0 \) not exceeding \( n \) and set

\[
S_n^{b_n} := \sum_{i=b_n+1}^{n} X_{\pi(i)}.
\]
We might obtain a strong law of large numbers after trimming. Let \( \pi \in S_n \) be pointwise defined such that

\[
X_{\pi(1)} \geq \ldots \geq X_{\pi(n)}
\]
is a rearrangement of \( X_1, \ldots, X_n \).

In some literature the order statistics is also denoted by

\[
X_{n,1} \geq \ldots \geq X_{n,n}.
\]

For the following let \((b_n)\) always be a sequence of numbers in \( \mathbb{N}_0 \) not exceeding \( n \) and set

\[
S_{n}^{b_n} := \sum_{i=b_n+1}^{n} X_{\pi(i)}.
\]

**Definition 2.1**

The sum \( S_{n}^{b_n} \) is called

- **lightly trimmed sum** if \( b_n = r \in \mathbb{N} \),
- **intermediately (moderately) trimmed sum** if \( \lim_{n \to \infty} b_n = \infty \) and \( b_n = o(n) \),
- **heavily trimmed sum** if \( b_n \sim \kappa \cdot n, \ 0 < \kappa < 1 \).
Let us first look at the example \( F(x) = 1 - 1/x \).
Let us first look at the example $F(x) = 1 - 1/x$.
Then $\mathbb{E}(X_1) = \infty$ and we don’t obtain a strong law of large numbers.
Let us first look at the example $F(x) = 1 - 1/x$.

Then $E(X_1) = \infty$ and we don’t obtain a strong law of large numbers.

If $(X_n)$ are i.i.d. it follows from [Kesten and Maller, 1992, Application of Theorem 2.3] that

$$\lim_{n \to \infty} \frac{S_n^1}{n \log n} = 1 \text{ a.s.}$$
Let us first look at the example $F(x) = 1 - 1/x$.

Then $E(X_1) = \infty$ and we don’t obtain a strong law of large numbers.

If $(X_n)$ are i.i.d. it follows from [Kesten and Maller, 1992, Application of Theorem 2.3] that

$$\lim_{n \to \infty} \frac{S_n^1}{n \log n} = 1 \text{ a.s.}$$

If $(X_n)$ are sufficiently fast $\psi$-mixing it follows from [Aaronson and Nakada, 2003, Application of Theorem 1.1] that

$$\lim_{n \to \infty} \frac{S_n^1}{n \log n} = 1 \text{ a.s.}$$
But which dynamical systems are $\psi$-mixing?
But which dynamical systems are $\psi$-mixing?
We give the probably first example of a dynamical system proving strong laws of large numbers under trimming.
But which dynamical systems are \( \psi \)-mixing?
We give the probably first example of a dynamical system proving strong laws of large numbers under trimming.
Consider the unique continued fraction expansion of an irrational \( x \in [0, 1] \) given by
\[
x := [a_1(x), a_2(x), \ldots] := \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots}}.
\]
But which dynamical systems are $\psi$-mixing?
We give the probably first example of a dynamical system proving strong laws of large numbers under trimming.
Consider the unique continued fraction expansion of an irrational $x \in [0, 1]$ given by
\[
x := [a_1(x), a_2(x), \ldots] := \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots}}.
\]

We consider $a_1, a_2, \ldots$ as random variables.
But which dynamical systems are $\psi$-mixing?
We give the probably first example of a dynamical system proving strong laws of large numbers under trimming.
Consider the unique continued fraction expansion of an irrational $x \in [0, 1]$ given by

$$x := [a_1(x), a_2(x), \ldots] := \frac{1}{a_1(x) + \frac{1}{a_2(x) + \ldots}}.$$ 

We consider $a_1, a_2, \ldots$ as random variables. Define

$$\phi : [0, 1] \rightarrow \mathbb{R}_{>0}, \quad x \mapsto \lfloor 1/x \rfloor$$

and

$$T : [0, 1] \rightarrow [0, 1], \quad x \mapsto 1/x \mod 1.$$
But which dynamical systems are $\psi$-mixing?
We give the probably first example of a dynamical system proving strong laws of large numbers under trimming.
Consider the unique continued fraction expansion of an irrational $x \in [0, 1]$ given by $x := [a_1(x), a_2(x), \ldots] := \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots}}$.

We consider $a_1, a_2, \ldots$ as random variables. Define

$\phi: [0, 1] \to \mathbb{R}_{>0}$

$x \mapsto \lfloor 1/x \rfloor$

Then $a_n(x) = \phi \circ T^{n-1}(x)$. 
The example $P(X_1 > x) = L(x)/x$

Lemma 2.2 ([Diamond and Vaaler, 1986])

If $X_n := a_n$, $n \in \mathbb{N}$, then we have that

$$\lim_{n \to \infty} \frac{S_n^1}{n \log n} = \frac{1}{\log 2} \text{ a.s.}$$

(with respect to Lebesgue or $\gamma$, the invariant measure with respect to the Gauss system.)
The example \( \mathbb{P}(X_1 > x) = L(x)/x \)

Lemma 2.2 ([Diamond and Vaaler, 1986])

If \( X_n := a_n, \ n \in \mathbb{N}, \) then we have that

\[
\lim_{n \to \infty} \frac{S_n^1}{n \log n} = \frac{1}{\log 2} \text{ a.s.}
\]

(with respect to Lebesgue or \( \gamma \), the invariant measure with respect to the Gauss system.)

The continued fraction digits are a special example of exponentially fast \( \psi \)-mixing random variables and the results by Aaronson and Nakada could be applied.
Lemma 2.2 ([Diamond and Vaaler, 1986])

If $X_n := a_n$, $n \in \mathbb{N}$, then we have that

$$\lim_{n \to \infty} \frac{S_n^1}{n \log n} = \frac{1}{\log 2} \text{ a.s.}$$

(with respect to Lebesgue or $\gamma$, the invariant measure with respect to the Gauss system.)

The continued fraction digits are a special example of exponentially fast $\psi$-mixing random variables and the results by Aaronson and Nakada could be applied. However, $\psi$-mixing is a strong condition on dynamical systems and not all interesting dynamical systems are $\psi$-mixing...
... and light trimming is not always enough, either!
... and light trimming is not always enough, either! Define

\[ \phi : [0, 1] \rightarrow \mathbb{R}_{>0}, \]
\[ x \mapsto \left\lfloor \frac{1}{x} \right\rfloor \]

\[ \tilde{T} : [0, 1] \rightarrow [0, 1] \]
\[ x \mapsto 2x \mod 1. \]
... and light trimming is not always enough, either! Define

\[
\phi: [0, 1] \to \mathbb{R}_{>0}, \quad x \mapsto \lfloor 1/x \rfloor
\]

\[
\tilde{T}: [0, 1] \to [0, 1] \quad x \mapsto 2x \mod 1.
\]

**Theorem 2.3 ([Haynes, 2014, Theorem 4, generalized])**

If \( X_n = \phi \circ \tilde{T}^{n-1} \), then for all positive valued sequences \((d_n)_{n \in \mathbb{N}}\) and \( k \in \mathbb{N} \) we have (with respect to the Lebesgue measure \( \lambda \)) that either

\[
\limsup_{n \to \infty} \frac{S_n^k}{d_n} = \infty \text{ a.s. or } \liminf_{n \to \infty} \frac{S_n^k}{d_n} = 0 \text{ a.s.}
\]
Strong laws under trimming - a comparison between iid random variables and dynamical systems

The example $P(X_1 > x) = L(x)/x$

Comparison: Continued fractions (left) / doubling map (right)

Main difference: The observable $\phi$ obeys the structure of the underlying dynamics $T$ but not of $\tilde{T}$. If $\phi \circ \tilde{T}^n > 1$ then $\phi \circ \tilde{T}^{n+1} = \lfloor \phi \circ \tilde{T}^{n+1} / 2 \rfloor$. So let's try intermediate trimming!
Comparison: Continued fractions (left) / doubling map (right)

Main difference: The observable $\phi$ obeys the structure of the underlying dynamics $T$ but not of $\tilde{T}$. 

\[ \mathbb{P}(X_1 > x) = \frac{L(x)}{x} \]
Comparison: Continued fractions (left) / doubling map (right)

Main difference: The observable $\phi$ obeys the structure of the underlying dynamics $T$ but not of $\tilde{T}$.

If $\phi \circ \tilde{T}^n > 1$ then $\phi \circ \tilde{T}^{n+1} = \left\lfloor \frac{\phi \circ \tilde{T}^{n+1}}{2} \right\rfloor$. 
Comparison: Continued fractions (left) / doubling map (right)

Main difference: The observable \( \phi \) obeys the structure of the underlying dynamics \( T \) but not of \( \tilde{T} \).

If \( \phi \circ \tilde{T}^n > 1 \) then \( \phi \circ \tilde{T}^{n+1} = \left\lfloor \frac{\phi \circ \tilde{T}^{n+1}}{2} \right\rfloor \).

So let's try intermediate trimming!
We define $\Psi := \left\{ u : \mathbb{N} \to \mathbb{R}_{>0} : \sum_{n=1}^{\infty} \frac{1}{u(n)} < \infty \right\}$. 
We define \( \Psi := \left\{ u : \mathbb{N} \to \mathbb{R}_{>0} : \sum_{n=1}^{\infty} \frac{1}{u(n)} < \infty \right\} \).

**Theorem 2.4 ([S., 2018, Theorem 1.1 & 1.2])**

Let \( (X_n) = (\phi \circ \tilde{T}^{n-1}) \) and let \( \lim_{n \to \infty} b_n / \log^{1/4} n = 0 \). If there exists \( \psi \in \Psi \) such that

\[
b_n := \left\lfloor \frac{\log \psi (\lfloor \log n \rfloor) - \log \log n}{\log 2} \right\rfloor,
\]

then there exists a norming sequence \( (d_n) \) such that

\[
\lim_{n \to \infty} \frac{S_{b_n}^{d_n}}{d_n} = 1 \text{ a.s.} \tag{1}
\]

In case that (1) holds we have that \( d_n = n \cdot \log n \).
Strong laws under trimming - a comparison between iid random variables and dynamical systems

The example $P(X_1 > x) = L(x)/x$

We define $\Psi := \left\{ u : \mathbb{N} \to \mathbb{R}_{>0} : \sum_{n=1}^{\infty} \frac{1}{u(n)} < \infty \right\}$.

**Theorem 2.4 ([S., 2018, Theorem 1.1 & 1.2])**

Let $(X_n) = (\phi \circ \tilde{T}^{n-1})$ and let $\lim_{n \to \infty} b_n / \log^{1/4} n = 0$. Iff there exists $\psi \in \Psi$ such that

$$b_n := \left\lfloor \frac{\log \psi (\lfloor \log n \rfloor) - \log \log n}{\log 2} \right\rfloor,$$

then there exists a norming sequence $(d_n)$ such that

$$\lim_{n \to \infty} \frac{S_{b_n}^{n}}{d_n} = 1 \text{ a.s.} \quad (1)$$

In case that (1) holds we have that $d_n = n \cdot \log n$.

In particular we have that $b_n = \lfloor u \cdot \log \log \log n \rfloor$ for all $n \in \mathbb{N}$ is a possible trimming sequence if and only if $u > 1/\log 2$. 
Strong laws under trimming - a comparison between iid random variables and dynamical systems

The example $\mathbb{P}(X_1 > x) = L(x)/x$

We define $\Psi := \left\{ u : \mathbb{N} \rightarrow \mathbb{R}_{>0} : \sum_{n=1}^{\infty} \frac{1}{u(n)} < \infty \right\}$.

**Theorem 2.4 ([S., 2018, Theorem 1.1 & 1.2])**

Let $(X_n) = (\phi \circ \tilde{T}^{n-1})$ and let $\lim_{n \to \infty} b_n / \log^{1/4} n = 0$. Iff there exists $\psi \in \Psi$ such that

$$b_n := \left\lfloor \frac{\log \psi (\lfloor \log n \rfloor) - \log \log n}{\log 2} \right\rfloor,$$

then there exists a norming sequence $(d_n)$ such that

$$\lim_{n \to \infty} \frac{S_{b_n}^{d_n}}{d_n} = 1 \text{ a.s.} \tag{1}$$

In case that (1) holds we have that $d_n = n \cdot \log n$.

In particular we have that $b_n = \lfloor u \cdot \log \log \log n \rfloor$ for all $n \in \mathbb{N}$ is a possible trimming sequence if and only if $u > 1/\log 2$.

It is work in progress to determine limit laws for more general settings than the doubling map and the observable $\phi$. 
What if we consider other non-negative random variables such that
\[ P(X_1 > x) = \frac{L(x)}{x} \] with \( L \) a slowly varying function?
What if we consider other non-negative random variables such that 
\[ P(X_1 > x) = \frac{L(x)}{x} \] with \( L \) a slowly varying function?

\( \frac{L(\kappa \cdot x)}{L(x)} \) is called \textit{slowly varying} in infinity if for all \( \kappa > 0 \)

\[
\lim_{x \to \infty} \frac{L(\kappa \cdot x)}{L(x)} = 1.
\]
What if we consider other non-negative random variables such that
\( P(X_1 > x) = \frac{L(x)}{x} \) with \( L \) a slowly varying function?

\( L : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is called \textit{slowly varying} in infinity if for all \( \kappa > 0 \)

\[
\lim_{x \rightarrow \infty} \frac{L(\kappa \cdot x)}{L(x)} = 1.
\]

[Kesten and Maller, 1992, Theorem 2.3] gives a minimal trimming number for a lightly trimmed strong law depending on the distribution function.
What if we consider other non-negative random variables such that
\( P(X_1 > x) = \frac{L(x)}{x} \) with \( L \) a slowly varying function?

\( L : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is called *slowly varying* in infinity if for all \( \kappa > 0 \)

\[
\lim_{x \to \infty} \frac{L(\kappa \cdot x)}{L(x)} = 1.
\]

- [Kesten and Maller, 1992, Theorem 2.3] gives a minimal trimming number for a lightly trimmed strong law depending on the distribution function.

- [Aaronson and Nakada, 2003, Theorem 1.1] gives a generalisation for sufficiently fast \( \psi \)-mixing random variables under the same conditions on the distribution function as in the i.i.d. case.
What if we consider other non-negative random variables such that $\mathbb{P}(X_1 > x) = L(x)/x$ with $L$ a slowly varying function? $L : \mathbb{R}^+ \to \mathbb{R}^+$ is called *slowly varying* in infinity if for all $\kappa > 0$

$$
\lim_{x \to \infty} \frac{L(\kappa \cdot x)}{L(x)} = 1.
$$

- [Kesten and Maller, 1992, Theorem 2.3] gives a minimal trimming number for a lightly trimmed strong law depending on the distribution function.
- [Aaronson and Nakada, 2003, Theorem 1.1] gives a generalisation for sufficiently fast $\psi$-mixing random variables under the same conditions on the distribution function as in the i.i.d. case.
- If for example $F(x) = 1 - \exp(-\log^{3/2}(x))$, then there exists $(d_n)$ such that $\lim_{n \to \infty} \frac{S_n^2}{d_n} = 1$ a.s. while deleting only one digit does not work.
What if we consider other non-negative random variables such that
\[ P(X_1 > x) = \frac{L(x)}{x} \]
with \( L \) a slowly varying function?

\( L : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is called \textit{slowly varying} in infinity if for all \( \kappa > 0 \)

\[
\lim_{x \to \infty} \frac{L(\kappa \cdot x)}{L(x)} = 1.
\]

- [Kesten and Maller, 1992, Theorem 2.3] gives a minimal trimming number for a lightly trimmed strong law depending on the distribution function.
- [Aaronson and Nakada, 2003, Theorem 1.1] gives a generalisation for sufficiently fast \( \psi \)-mixing random variables under the same conditions on the distribution function as in the i.i.d. case.
- If for example \( F(x) = 1 - \exp\left(-\log^{3/2}(x)\right) \), then there exists \( (d_n) \)
such that \( \lim_{n \to \infty} \frac{S_n^2}{d_n} = 1 \) a.s. while deleting only one digit does not work.
- But there are also distribution functions \( F(x) = 1 - \frac{L(x)}{x} \) such that there is no strong law of large numbers under light trimming, one example is given in [Aaronson and Nakada, 2003].
What if we consider other non-negative random variables such that  
\( P(X_1 > x) = \frac{L(x)}{x} \) with \( L \) a slowly varying function? 

\( L : \mathbb{R}^+ \to \mathbb{R}^+ \) is called \textit{slowly varying} in infinity if for all \( \kappa > 0 \)

\[
\lim_{x \to \infty} \frac{L(\kappa \cdot x)}{L(x)} = 1.
\]

- [Kesten and Maller, 1992, Theorem 2.3] gives a minimal trimming number for a lightly trimmed strong law depending on the distribution function.
- [Aaronson and Nakada, 2003, Theorem 1.1] gives a generalisation for sufficiently fast \( \psi \)-mixing random variables under the same conditions on the distribution function as in the i.i.d. case.
- If for example \( F(x) = 1 - \exp\left(-\log^{3/2}(x)\right) \), then there exists \( (d_n) \) such that  
  \[
  \lim_{n \to \infty} \frac{S_n^2}{d_n} = 1 \text{ a.s. while deleting only one digit does not work.}
  \]
- But there are also distribution functions \( F(x) = 1 - \frac{L(x)}{x} \) such that there is no strong law of large numbers under light trimming, one example is given in [Aaronson and Nakada, 2003].
- To my knowledge for such functions not more is known neither in the i.i.d. nor in the dynamical systems setting.
1. The finite expectation case

2. The example $\mathbb{P}(X_1 > x) = L(x)/x$

3. The example $\mathbb{P}(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$
   - The i.i.d. case
   - The dynamical systems case
   - Mean convergence
Let us now assume $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying, $0 < \alpha < 1$. 
Let us now assume $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying, $0 < \alpha < 1$.

Haeusler and Mason proved laws of the iterated logarithm for intermediately trimmed sums, see [Haeusler and Mason, 1987] and [Haeusler, 1993].
Let us now assume $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying, $0 < \alpha < 1$.

Haeusler and Mason proved laws of the iterated logarithm for intermediately trimmed sums, see [Haeusler and Mason, 1987] and [Haeusler, 1993]. From those we can conclude the following theorem.

**Theorem 3.1 ([Haeusler and Mason, 1987, Haeusler, 1993, Applications from])**

Let $(X_n)$ be a sequence of i.i.d. non-negative random variables such that $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying and $\alpha \in (0, 1)$. Further, let $(b_n) = o(n)$ a sequence of natural numbers.
The example $P(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$

The i.i.d. case

- Let us now assume $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying, $0 < \alpha < 1$.
- Haeusler and Mason proved laws of the iterated logarithm for intermediately trimmed sums, see [Haeusler and Mason, 1987] and [Haeusler, 1993]. From those we can conclude the following theorem.

**Theorem 3.1 ([Haeusler and Mason, 1987, Haeusler, 1993, Applications from])**

Let $(X_n)$ be a sequence of i.i.d. non-negative random variables such that $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying and $\alpha \in (0, 1)$. Further, let $(b_n) = o(n)$ a sequence of natural numbers.

If $\lim \inf_{n \to \infty} b_n / \log \log n = \infty$, then there exists a sequence $(d_n)$ such that

$$\lim_{n \to \infty} \frac{S_n^{b_n}}{d_n} = 1 \text{ a.s.} \quad (2)$$
The example $P(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$

The i.i.d. case

- Let us now assume $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying, $0 < \alpha < 1$.
- Haeusler and Mason proved laws of the iterated logarithm for intermediately trimmed sums, see [Haeusler and Mason, 1987] and [Haeusler, 1993]. From those we can conclude the following theorem.

Theorem 3.1 ([Haeusler and Mason, 1987, Haeusler, 1993, Applications from])

Let $(X_n)$ be a sequence of i.i.d. non-negative random variables such that $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying and $\alpha \in (0, 1)$. Further, let $(b_n) = o(n)$ a sequence of natural numbers. If $\lim \inf_{n \to \infty} b_n/ \log \log n = \infty$, then there exists a sequence $(d_n)$ such that

$$\lim_{n \to \infty} \frac{S_{b_n}^n}{d_n} = 1 \text{ a.s.}$$

If $\lim \sup_{n \to \infty} b_n/ \log \log n < \infty$, then there exists no sequence $(d_n)$ such that (2) holds.
The i.i.d. case

- Let us now assume $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying, $0 < \alpha < 1$.
- Haeusler and Mason proved laws of the iterated logarithm for intermediately trimmed sums, see [Haeusler and Mason, 1987] and [Haeusler, 1993]. From those we can conclude the following theorem.

Theorem 3.1 ([Haeusler and Mason, 1987, Haeusler, 1993, Applications from])

Let $(X_n)$ be a sequence of i.i.d. non-negative random variables such that $F(x) = 1 - L(x)/x^\alpha$ with $L$ slowly varying and $\alpha \in (0, 1)$. Further, let $(b_n) = o(n)$ a sequence of natural numbers.

If $\lim \inf_{n \to \infty} b_n/\log \log n = \infty$, then there exists a sequence $(d_n)$ such that

$$\lim_{n \to \infty} \frac{S_{bn}}{d_n} = 1 \text{ a.s.} \tag{2}$$

If $\lim \sup_{n \to \infty} b_n/\log \log n < \infty$, then there exists no sequence $(d_n)$ such that (2) holds.

The norming sequence $(d_n)$ can be given explicitly.
The example $\mathbb{P}(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$

The dynamical systems case

1. The finite expectation case

2. The example $\mathbb{P}(X_1 > x) = L(x)/x$

3. The example $\mathbb{P}(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$
   - The i.i.d. case
   - The dynamical systems case
   - Mean convergence
The example \( P(X_1 > x) = \frac{L(x)}{x^\alpha}, \alpha \in (0, 1) \)

The dynamical systems case

Consider two new systems: Define

\[
\chi: [0, 1] \rightarrow \mathbb{R}_{>0} \quad T: [0, 1] \rightarrow [0, 1], \quad \widetilde{T}: [0, 1] \rightarrow [0, 1],
\]

\[
x \mapsto \lfloor \frac{1}{x} \rfloor^2 \quad x \mapsto \frac{1}{x} \mod 1 \quad x \mapsto 2x \mod 1.
\]
Consider two new systems: Define

$$
\chi: [0, 1] \rightarrow \mathbb{R}_{>0} \quad T: [0, 1] \rightarrow [0, 1], \quad \tilde{T}: [0, 1] \rightarrow [0, 1],
$$

$$
x \mapsto \left\lfloor \frac{1}{x} \right\rfloor^2 \quad x \mapsto \frac{1}{x} \mod 1 \quad x \mapsto 2x \mod 1.
$$

For $\left( X_n \right) = \left( \chi \circ T^{n-1} \right)$ and for $\left( X_n \right) = \left( \chi \circ \tilde{T}^{n-1} \right)$ we obtain the same trimmed strong law:
The example $P(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$

The dynamical systems case

Theorem 3.2 (Application of [Kesseböhmer and S., 2018, Theorem 1.7])

Let $X_n$ be given as above. Further, let $(b_n)_{n \in \mathbb{N}}$ be a sequence of natural numbers tending to infinity with $b_n = o(n)$. If

$$\lim_{n \to \infty} \frac{b_n}{\log \log n} = \infty,$$

then there exists a positive valued sequence $(d_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \frac{S_n}{d_n} = 1 \text{ a.s.}$$

In this case $d_n \sim \frac{n^2}{b_n}$. 
The example \( \mathbb{P}(X_1 > x) = L(x)/x^\alpha, \alpha \in (0,1) \)

The dynamical systems case

**Theorem 3.2 (Application of [Kesseböhmer and S., 2018, Theorem 1.7])**

Let \( X_n \) be given as above. Further, let \( (b_n)_{n \in \mathbb{N}} \) be a sequence of natural numbers tending to infinity with \( b_n = o(n) \). If

\[
\lim_{n \to \infty} \frac{b_n}{\log \log n} = \infty,
\]

then there exists a positive valued sequence \( (d_n)_{n \in \mathbb{N}} \) such that

\[
\lim_{n \to \infty} \frac{S_n}{d_n} = 1 \text{ a.s.}
\]

In this case \( d_n \sim \frac{n^2}{b_n} \).

We note that in both cases \( (X_n) = (\chi \circ T^{n-1}) \) and \( (X_n) = (\chi \circ \tilde{T}^{n-1}) \) the condition on the norming sequence \( (b_n) \) is the same as in the i.i.d. case.
Indeed [Kesseböhmer and S., 2018] gives general conditions for dynamical systems fulfilling a spectral gap property on the transfer operator and observables with regularly varying tails with exponent strictly between 0 and 1 for a strong law under intermediate trimming to hold.
The example \( P(X_1 > x) = \frac{L(x)}{x^\alpha}, \alpha \in (0, 1) \)

The dynamical systems case

- Indeed [Kesseböhmer and S., 2018] gives general conditions for dynamical systems fulfilling a spectral gap property on the transfer operator and observables with regularly varying tails with exponent strictly between 0 and 1 for a strong law under intermediate trimming to hold.

- As an application we obtain strong laws under trimming for piecewise expanding interval maps.
Strong laws under trimming - a comparison between iid random variables and dynamical systems

The example $P(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$

The dynamical systems case

- Indeed [Kesseböhmer and S., 2018] gives general conditions for dynamical systems fulfilling a spectral gap property on the transfer operator and observables with regularly varying tails with exponent strictly between 0 and 1 for a strong law under intermediate trimming to hold.
- As an application we obtain strong laws under trimming for piecewise expanding interval maps.
- Another application of these results gives strong laws under trimming for subshifts of finite type, see [Kesseböhmer and S., 2019a].
1. The finite expectation case

2. The example \( P(X_1 > x) = L(x)/x \)

3. The example \( P(X_1 > x) = L(x)/x^\alpha, \alpha \in (0, 1) \)
   - The i.i.d. case
   - The dynamical systems case
   - Mean convergence
The strong law of large numbers or Birkhoff’s ergodic theorem give for i.i.d. or ergodic and identically distributed, integrable random variables

$$\lim_{n \to \infty} \frac{S_n}{\mathbb{E}(S_n)} = 1 \text{ a.s.}$$
The strong law of large numbers or Birkhoff’s ergodic theorem give for i.i.d. or ergodic and identically distributed, integrable random variables

\[
\lim_{n \to \infty} \frac{S_n}{\mathbb{E}(S_n)} = 1 \text{ a.s.}
\]

In the generalized case for non-integrable random variables we obtain a strong law after trimming, i.e. there exists a (possibly constant) sequence of natural numbers \((b_n)\) and a norming sequence \((d_n)\) fulfilling

\[
\lim_{n \to \infty} \frac{S_n^{b_n}}{d_n} = 1 \text{ a.s.} \quad (5)
\]
The strong law of large numbers or Birkhoff’s ergodic theorem give for i.i.d. or ergodic and identically distributed, integrable random variables

\[
\lim_{n \to \infty} \frac{S_n}{\mathbb{E}(S_n)} = 1 \text{ a.s.}
\]

In the generalized case for non-integrable random variables we obtain a strong law after trimming, i.e. there exists a (possibly constant) sequence of natural numbers \((b_n)\) and a norming sequence \((d_n)\) fulfilling

\[
\lim_{n \to \infty} \frac{S_n^{b_n}}{d_n} = 1 \text{ a.s.} \quad (5)
\]

Can we say anything about the norming sequence \((d_n)\)?
The strong law of large numbers or Birkhoff’s ergodic theorem give for i.i.d. or ergodic and identically distributed, integrable random variables

\[
\lim_{n \to \infty} \frac{S_n}{\mathbb{E}(S_n)} = 1 \text{ a.s.}
\]

In the generalized case for non-integrable random variables we obtain a strong law after trimming, i.e. there exists a (possibly constant) sequence of natural numbers \((b_n)\) and a norming sequence \((d_n)\) fulfilling

\[
\lim_{n \to \infty} \frac{S_{b_n}}{d_n} = 1 \text{ a.s. } (5)
\]

Can we say anything about the norming sequence \((d_n)\)?

In general: No!

Even for i.i.d. random variables there are examples for which (5) holds but \(\mathbb{E}(S_{b_n}) = \infty\), see [Kesseböhmer and S., 2019c, Remark 3].
The example $P(X_1 > x) = L(x)/x^\alpha, \alpha \in (0, 1)$

Mean convergence

However, remember the two systems from before:

$\chi: [0, 1] \to \mathbb{R}_{>0}$  \quad  $T: [0, 1] \to [0, 1]$,  \quad  $\tilde{T}: [0, 1] \to [0, 1]$,  

\begin{align*}
  x &\mapsto [1/x]^2  \\
  x &\mapsto 1/x \mod 1  \\
  x &\mapsto 2x \mod 1
\end{align*}
However, remember the two systems from before:

\( \chi : [0, 1] \to \mathbb{R}_0^+ \quad T : [0, 1] \to [0, 1], \quad \tilde{T} : [0, 1] \to [0, 1], \)

\( x \mapsto [1/x]^2 \quad x \mapsto 1/x \mod 1 \quad x \mapsto 2x \mod 1 \)

For \( X_n = \chi \circ T^{n-1} \) we have that \( d_n \sim \mathbb{E} (S_{n}^{b_n}) \).
However, remember the two systems from before:

\[
\chi: [0, 1] \rightarrow \mathbb{R}_{>0}, \quad T: [0, 1] \rightarrow [0, 1], \quad \tilde{T}: [0, 1] \rightarrow [0, 1],
\]

\[
\chi(x) = \left\lfloor \frac{1}{x} \right\rfloor^2, \quad x \mapsto \frac{1}{x} \mod 1, \quad x \mapsto 2x \mod 1.
\]

For \( X_n = \chi \circ T^{n-1} \) we have that \( d_n \sim \mathbb{E}(S_n^{b_n}) \).

But for \( X_n = \chi \circ \tilde{T}^{n-1} \) we have that \( \mathbb{E}(S_n^{b_n}) = \infty \).
The example $P(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$

Mean convergence

However, remember the two systems from before:

$\chi: [0, 1] \to \mathbb{R}^+ \quad T: [0, 1] \to [0, 1], \quad \tilde{T}: [0, 1] \to [0, 1],$

$\chi \circ T^{n-1}$ we have that $d_n \sim \mathbb{E}(S_n^{b_n}).$

But for $X_n = \chi \circ \tilde{T}^{n-1}$ we have that $\mathbb{E}(S_n^{b_n}) = \infty.$

For details see [Kesseböhmer and S., 2019b].
The example $P(X_1 > x) = L(x)/x^\alpha$, $\alpha \in (0, 1)$.

Mean convergence

- [Kesseböhmer and S., 2019b] gives general conditions for mean convergence for the case $F(x) = 1 - L(x)/x^\alpha$, $L$ slowly varying, $0 < \alpha < 1$. 

The example \( P(X_1 > x) = L(x)/x^\alpha, \alpha \in (0, 1) \)

Mean convergence

- [Kesseböhmer and S., 2019b] gives general conditions for mean convergence for the case \( F(x) = 1 - L(x)/x^\alpha \), \( L \) slowly varying, \( 0 < \alpha < 1 \).

- If \( (X_n) \) are either independent or \( X_n = f \circ T^{n-1} \) with \( f \) sufficiently regular and \( T \) a piecewise expanding interval map and \( (X_n) \) are exponentially fast \( \psi \)-mixing and additionally we have for \( (b_n) \) and \( (d_n) \) that

\[
\lim_{n \to \infty} \frac{S^{b_n}_n}{d_n} = 1 \text{ a.s.}
\]

then we also have mean convergence.
The example $\mathbb{P}(X_1 > x) = \frac{L(x)}{x^\alpha}, \alpha \in (0, 1)$

Mean convergence

- [Kesseböhmer and S., 2019b] gives general conditions for mean convergence for the case $F(x) = 1 - \frac{L(x)}{x^\alpha}$, $L$ slowly varying, $0 < \alpha < 1$.
- If $(X_n)$ are either independent or $X_n = f \circ T^{n-1}$ with $f$ sufficiently regular and $T$ a piecewise expanding interval map and $(X_n)$ are exponentially fast $\psi$-mixing and additionally we have for $(b_n)$ and $(d_n)$ that

$$\lim_{n \to \infty} \frac{S_{b_n}^{d_n}}{d_n} = 1 \text{ a.s.}$$

then we also have mean convergence.
- As seen before the $\psi$-mixing property is essential.
On the ergodic theory of non-integrable functions and infinite measure spaces. 

Trimmed sums for non-negative, mixing stationary processes. 

Proof of the ergodic theorem. 
*Proceedings of the National Academy of Sciences*, 17(12):656–660.

On sums of independent random variables with infinite moments and “fair” games. 

Estimates for partial sums of continued fraction partial quotients. 

A limit theorem for random variables with infinite moments. 

A nonstandard law of the iterated logarithm for trimmed sums. 

Laws of the iterated logarithm for sums of the middle portion of the sample. 

Quantitative ergodic theorems for weakly integrable functions. 

Strong laws of large numbers for intermediately trimmed Birkhoff sums of observables with infinite mean. 

Intermediately trimmed strong laws for Birkhoff sums on subshifts of finite type. 

Mean convergence for intermediately trimmed Birkhoff sums of observables with regularly varying tails. 

Strong laws of large numbers for intermediately trimmed sums of i.i.d. random variables with infinite mean. 
Ratios of Trimmed Sums and Order Statistics.

Trimmed sums for observables on the doubling map.