Alex Kontorovich: Thin groups
7th Bremen Summer School and Symposium, August 2019

Q: Is this thin?

(a) $\Gamma = \left\langle \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{=:T}, \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=:S} \right\rangle$

$\leq \underbrace{SL_2(\mathbb{Z})}_{\text{arithmetic group}}$

exkurs: algebraic groups

$$SL_2 = \{ (a,b,c,d) \in A^4 \mid \underbrace{ad-bc-1}_{\text{polynomial in entries}} = 0 \}$$

A ring

+ group structure; operations given by polynomial functions

this is an example of an algebraic group

$SL_2(\mathbb{R}) \curvearrowright \mathbb{H} = \{ x+iy \in \mathbb{C} \mid y > 0 \}$. Let $G := SL_2(\mathbb{R})$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $z \in \mathbb{H}$:

$$g.z = \frac{az+b}{cz+d} \qquad \text{fractional linear transformation}$$

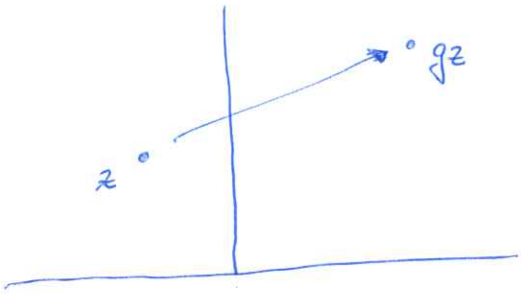
Exercise:

(1) is an action:

* is $g.z \in \mathbb{H}$?
* $g_1(g_2 z) = (g_1 g_2).z$?
* $id.z = z$ (ok, have shown this)

(2) Is $G \curvearrowright \mathbb{H}$ transitive? Yes, prove this

(3) $G \curvearrowright \mathbb{H}$ not simply transitive



We want a <u>fundamental domain</u> for $\Gamma \curvearrowright \mathbb{H}$, i.e.,

subset $\mathcal{F} \subseteq \mathbb{H}$ open s.th.

(1) $\forall z \in \mathbb{H} \; \exists \gamma \in \Gamma : \gamma z \in \overline{\mathcal{F}}$

(2) If $z, w \in \mathcal{F}$ and $\gamma \in \Gamma$ with $\gamma z = w$, then $z = w$.

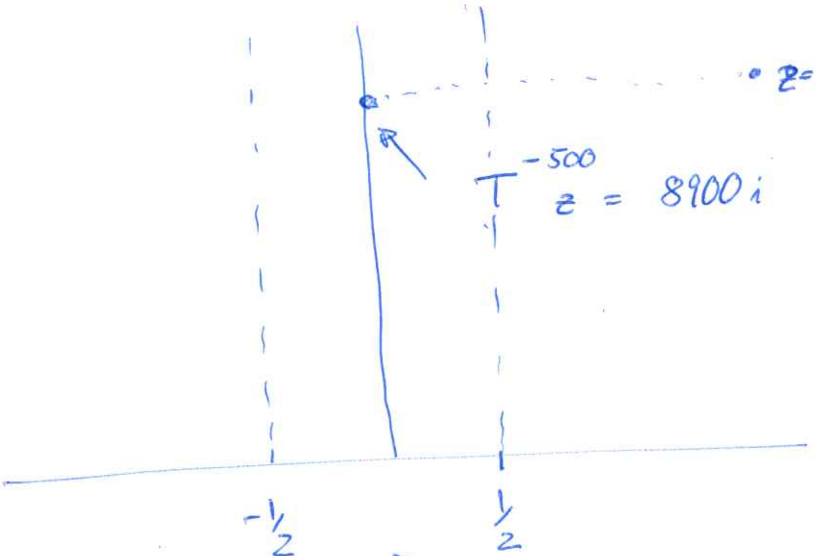
Try to find a fundamental domain:

$$T.z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . z = \frac{1 \cdot z + 1}{0 \cdot z + 1} = z + 1$$

$\mathbb{H}$

$z = 500 + 8900i$

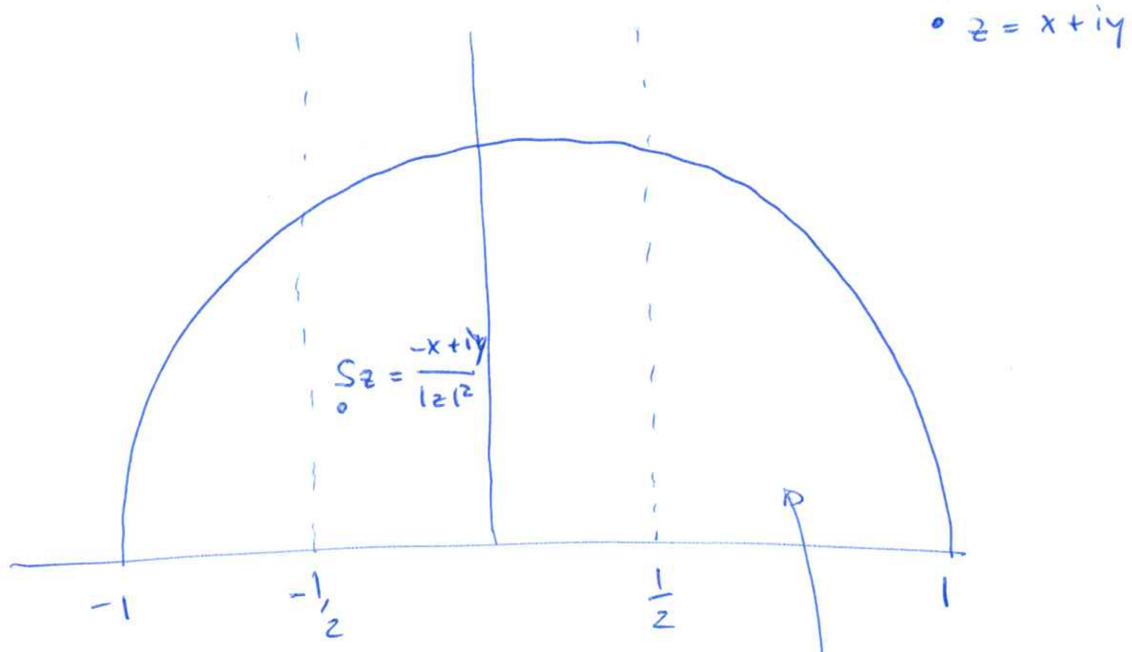
$T^{-500} z = 8900i$ (use T's to move near $i\mathbb{R}_{>0}$)



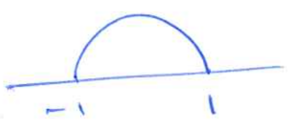
can move every point in here (possibly on boundary)

$$S.z = \frac{0.z+1}{-1\cdot z+0} = -\frac{1}{z} = -\frac{\overline{z}}{|z|^2} = \frac{-x+iy}{|z|^2}$$

$z = x+iy$

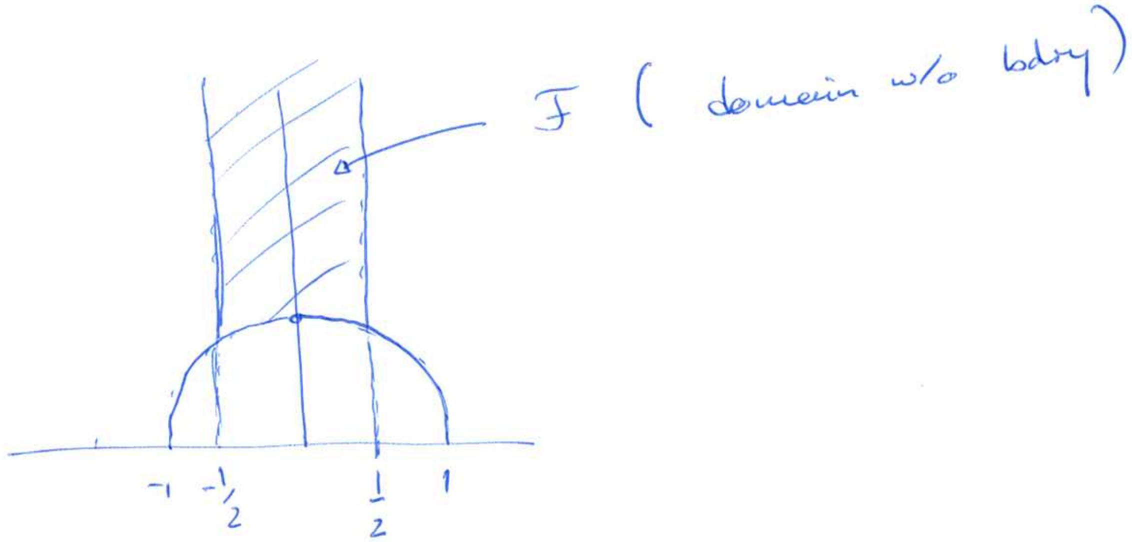


Can move every point from in here to out of

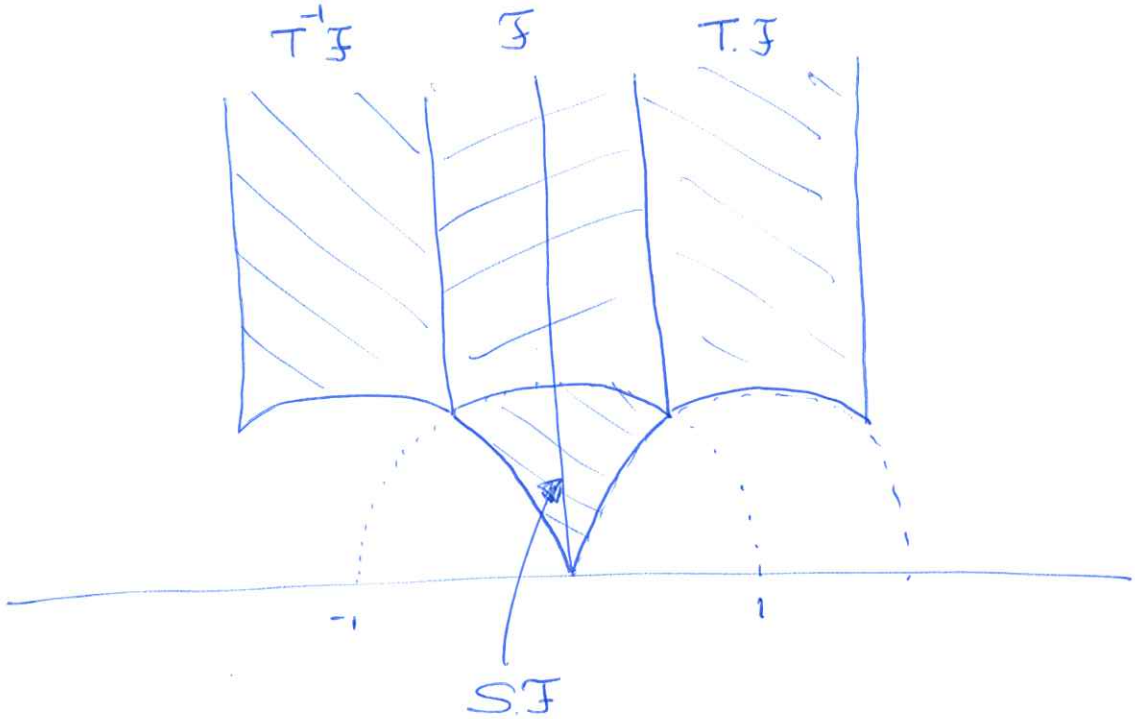


Have used up all possible moves

Get fundamental domain:



Rmk: ~~[illegible]~~ contruction has relation to continued fraction algorithm.



How to figure out images of $\mathcal{F}$ under Γ?

→ Look at vertices/ of $\mathcal{F}$;
boundary

$$\partial \mathcal{F} \subseteq \underbrace{[-\tfrac{1}{2}, \infty)}_{} \cup \underbrace{[\tfrac{1}{2}, \infty)}_{} \cup \underbrace{(-1, 1)}_{}$$

geodesic segments

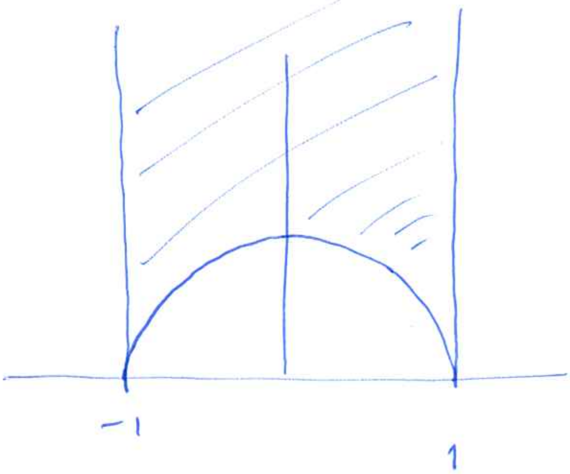
look at images of these geodesic segments.
Only need to under-stand where endpoints of geodesics go b/c we know how geodesics ~~in between~~ connecting points look like.

$$\partial S\mathcal{F} \subseteq [2, 0) \cup (-2, 0) \cup (1, -1).$$

Thus, $\Gamma = SL_2(\mathbb{Z})$. Not thin, but arithmetic.

(b) $\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \leq SL_2(\mathbb{Z})$

a fundamental domain:



Exercise:

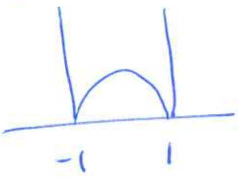
1) $d\mu = \dfrac{dx\,dy}{y^2}$ is G-invariant

2) $\mu(\mathcal{F}) = ?$ Discussed: is finite

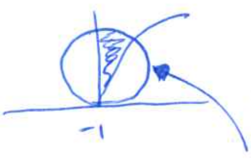
$\uparrow$ from (a)

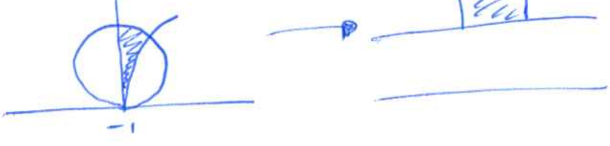
Now: is fund dom for the new Γ of finite volume?

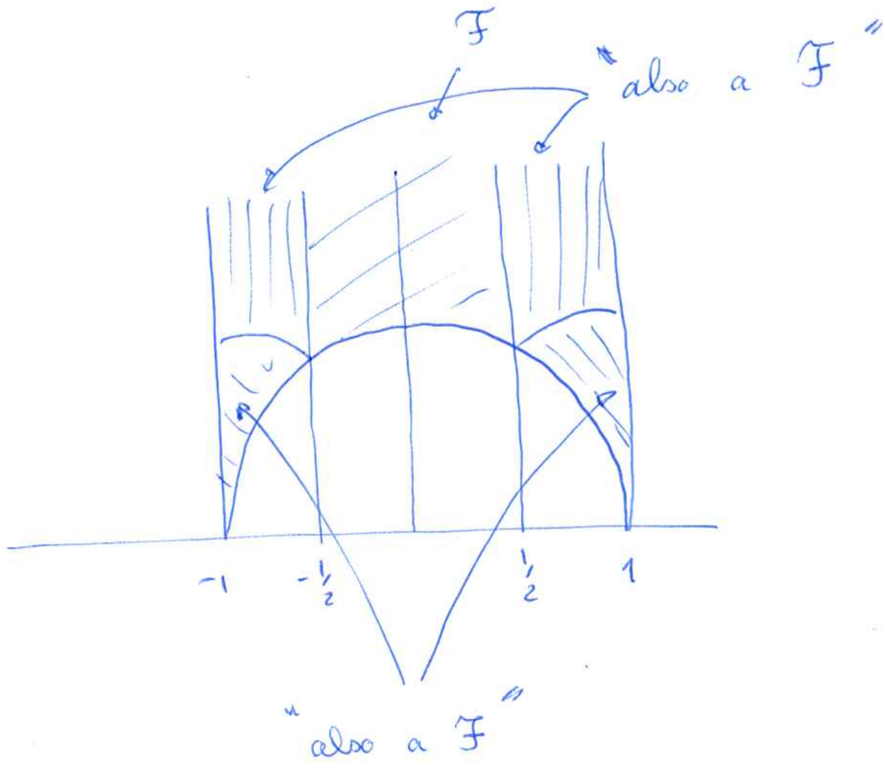
Idea: Understand how many copies of $\mathcal{F}$ are needed to

Build  :

Exercise: Find $g : -1 \longrightarrow \infty$ and see what happens

to 

Prove: 

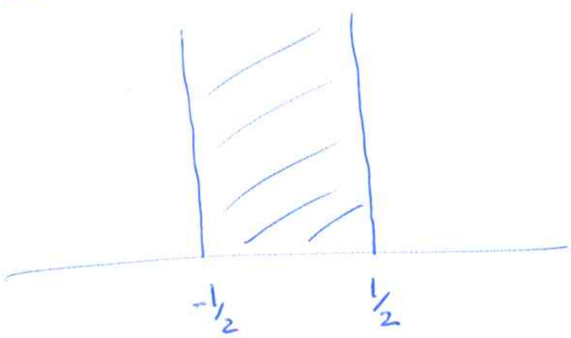


$\Rightarrow$ need 3 copies of $\mathcal{F}$ to build new fund dom
(+ cut and paste)

$\Rightarrow \Gamma$ has index 3 in $SL_2(\mathbb{Z})$. H is a congruence group.

(c) $\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \leq SL_2(\mathbb{Z})$

a fundamental domain:



Is this thin? No, because it is „cheaty".

Zariski closure

$$Zcl(\Gamma) = \left\{ (a,b,c,d) \;\middle|\; \begin{array}{l} \forall P \text{ polynomial } \in \{P \mid P(a,b,c,d)|_\Gamma = 0\}: \\ P(a,b,c,d) = 0 \end{array} \right\}$$

here:

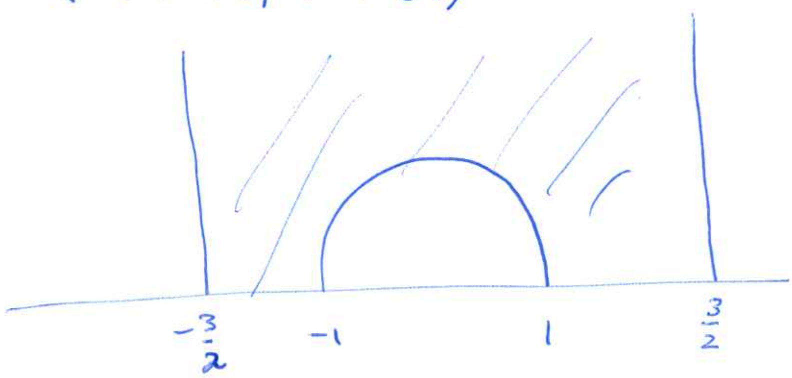
$$Zcl(\Gamma) = \mathbb{U} = \left\{ (a,b,c,d) \;\middle|\; \begin{array}{l} ad-bc-1=0, \quad c=0, \\ a-1=0, \quad d-1=0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \quad \text{the unipotent group}$$

Not thin because $\Gamma = \mathbb{U}(\mathbb{Z})$.

<u>Def</u>: Let $\gamma_1, \ldots, \gamma_k \in GL_n(\mathbb{Z})$. Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle$.
Let $G = Zcl(\Gamma)$. Then Γ is <u>thin</u> if $[G(\mathbb{Z}) : \Gamma] = \infty$.

(d) $\Gamma = \left\langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$



$[SL_2(\mathbb{Z}) : \Gamma] = \infty$. This is thin.

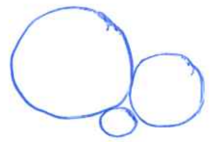
Fact: algebraic subgrps of SL_2:

$$\mathcal{U}, \quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad A = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

Reasons to care about thin groups

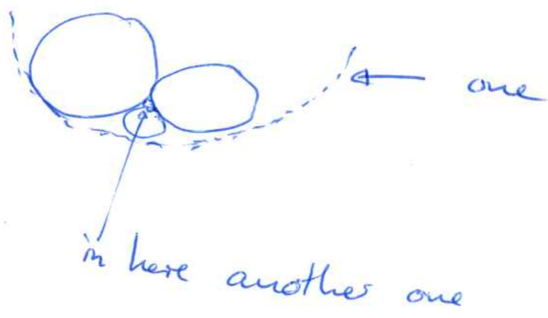
1) Apollonian Circle Packing:

Thm (Apollonius):



Given three tangent circles, construct (straight edge/compass) all possible circles tangent to given ones.

$\exists$ 2 such:

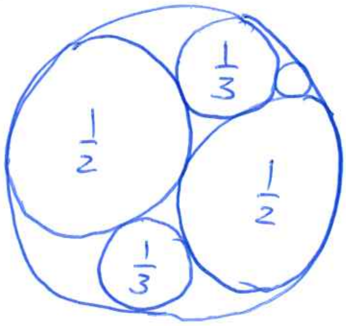


Exercise:
Find fastest possible solution to this problem
(how many times you have to draw something)
(fastest known: Boragar – K. '18)

Leibniz: iterate

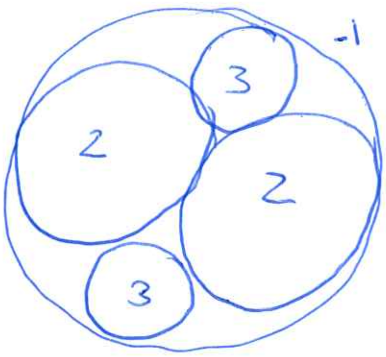
Soddy 1930s: wanted to pack cylinders:

radii



Integral Apollonian Circle Packings (IACP)

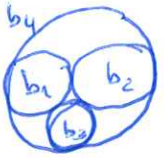
use bend $b(C) = \frac{1}{r(C)}$



IACP: configuration with <u>all</u> bends $\in \mathbb{Z}$

<u>Soddy</u>: $\exists$

Descartes (1630s):

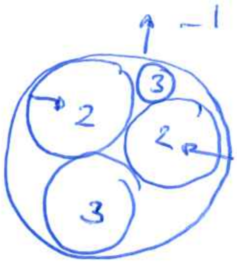
If  then $\sum b_j^2 = \frac{1}{2}\left(\sum b_j\right)^2$

Exercise: proof (inversive geometry)

test: $b_1 = 2$, $b_2 = 2$, $b_3 = 1$, $b_4 = ?$

$$0 = 2\left(\sum b_j^2\right) - \left(\sum b_j\right)^2$$

Need oriented bends!



$\rightsquigarrow$ $b_3 = -1$

then two solutions for b_4, both $= 3$.

Exercise: Given b_1, b_2, b_3,

$$b_4 + b_4' = 2(b_1 + b_2 + b_3)$$

$$\Rightarrow \quad b_4' = 2(b_1 + b_2 + b_3) - b_4$$

$\rightarrow$ all solutions are integers

Given IACP $\mathcal{P}$ (from above), let

$$\mathcal{B} = \{ b(C) \mid C \in \mathcal{P} \} = \{-1, 2, 3, 6, 14, \dots \}$$

Graham - Lagarias - Mallows - Wilks - Yau '03:

Local-global conjecture:

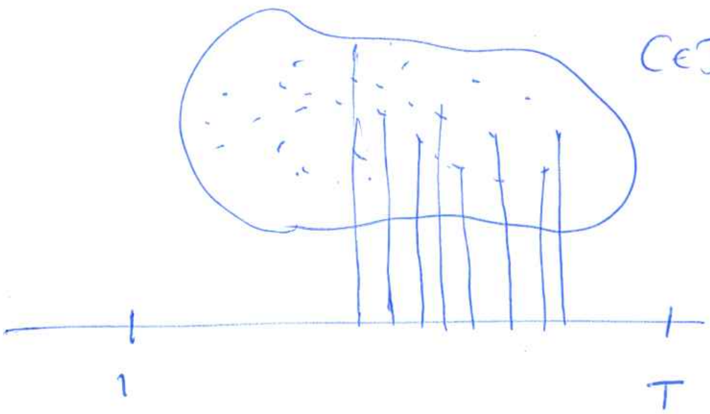
Every sufficiently large admissible integer is a bend in $\mathcal{P}$.

(here: prime $\mathcal{P}$; rmk: suff large and admissible depends on $\mathcal{P}$)

$$\#\{C\in\mathcal{P} \mid b(C)<T\} \underset{\text{K.-Oh'08}}{\sim} c_{\mathcal{P}}\, T^{\dim_H(\mathcal{P})}$$

$$\left(\underset{\underset{\delta}{\parallel}}{\dim_H}(\mathcal{P}) = 1.305\ldots\right)$$

Vinogradov, Lee-Oh $\Big\}$ $\#\{\ldots\} = c_{\mathcal{P}}\, T^{\delta} + O(T^{\delta-\varepsilon})$



$C\in\mathcal{P}$ $\quad \# \sim T^{1.3}$

Roughly, each fiber $\overset{?}{\approx} \dfrac{T^{1.3}}{T} = T^{0.3}$

Turns out: $\mathcal{B} \pmod{24} = \{0, 2, 3, 6, 14, 23\}$

<u>Def:</u> $n\in\mathbb{Z}$ is <u>admissible</u> if $n \equiv$ $\quad \bmod 24$

Known (GLMWY, Sarnak, Fuchs, Bourgain-Fuchs)

<u>Thm</u> (Bourgain-K.):

$$\exists\, \eta>0: \quad \frac{\#\mathcal{B}\cap[1,X]}{\#\underset{\text{admissible}}{\mathcal{A}}\cap[1,X]} = 1 + O(X^{-\eta}) \quad \text{as } X\to\infty$$

Where is the group?

at $b_4' = 2(b_1+b_2+b_3) - b_4$:

$$\underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}}_{=:\, S_4} \begin{pmatrix} b_1 \\ \vdots \\ b_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4' \end{pmatrix}$$

$$\Gamma := \langle S_1, S_2, S_3, S_4 \rangle \leq GL_4(\mathbb{Z})$$

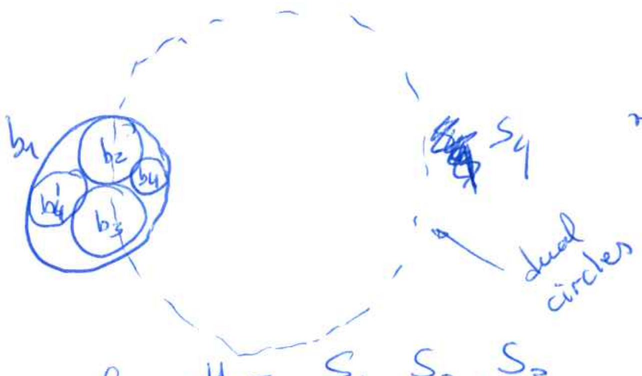
$$Zcl(\Gamma) = O_F$$

$$F(b) := 2 \sum b_j^2 - \left(\sum b_j\right)^2 \qquad (\text{see above})$$

$$\{\underset{\substack{\cap \\ GL_4(\mathbb{R})}}{g} \mid F(gb) = F(b) \text{ for all } b \in \mathbb{R}^4\} = O_F(\mathbb{R})$$

$O_F(\mathbb{Z})$ arithmetic group , $[O_F(\mathbb{Z}) : \Gamma] = \infty$

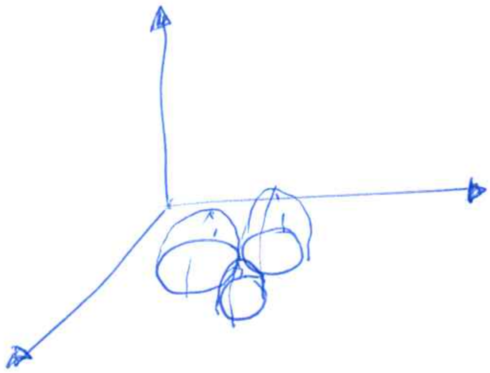
How can one see that Γ is a thin group?



reflection: $b_1 \to b_1$
$b_2 \to b_2$
$b_3 \to b_3$
$b_4 \to b_4'$

same for other S_1, S_2, S_3

$\hat{\mathbb{C}} = \partial \mathbb{H}^3$, $\qquad \mathbb{H}^3 = \{ (x,y,z) \mid z > 0 \}$



$$d\mu = \frac{dx\,dy\,dz}{z^3}$$

$SL_2(\mathbb{C}) \curvearrowright \mathbb{H}^3$

$\Gamma \backslash \mathbb{H}^3$

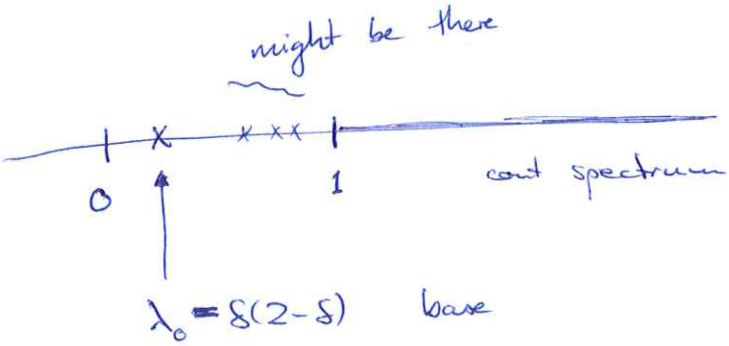
Congruence groups:

$$\Gamma(q) := \{ \gamma \in \Gamma \mid \gamma \equiv I \mod q \}$$

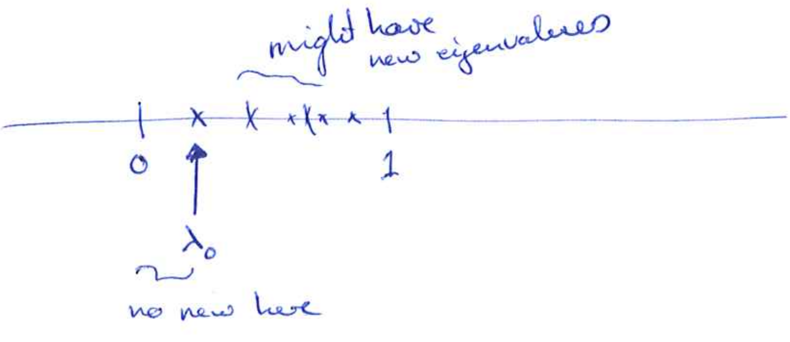
$[\Gamma : \Gamma(q)] < \infty \qquad L^2(\Gamma(q) \backslash \mathbb{H}^3)$

spec Δ:

Γ:



$\Gamma(q)$:



a priori:
new eigenvalues can go to λ_0 for $q \to \infty$

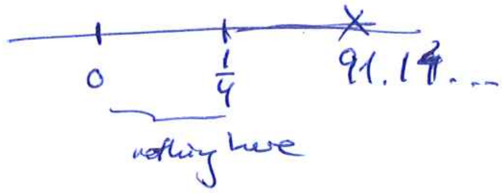
superapproximation: $\exists \eta > 0 \; \forall q$: no new eigenvalues below $\lambda_0 + \eta$

look at $\mathrm{Cay}\left(\Gamma / \Gamma(q), (S_1, \ldots, S_4)\right)$:

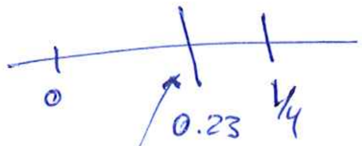
superapprox $\Longleftrightarrow$ get family of expanders

Ramanujan / Selberg:

$\underline{SL_2(\mathbb{Z})}$: spec Δ :



spec $\Delta_{\Gamma(q)}$:



Kim - Sarnak (bounds towards Selberg / Ramanujan)

McMullen classical arithm chaos conj (AC):

$$\# \Big\{ \underset{\underset{\alpha}{\shortparallel}}{[\overline{a_0; a_1, \ldots, a_k}]} \in \mathbb{Q}(\sqrt{5}) \;\Big|\; a_j \in \{1, 2\} \Big\} > c^k \;?$$

Local - Global Conj (Bourgain - K.)

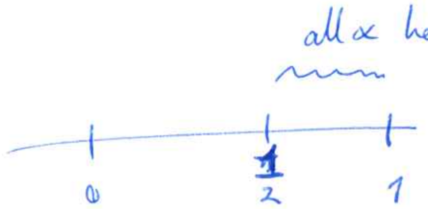
$$\Gamma = \Big\langle \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{=: S_1}, \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}}_{=: S_2} \Big\rangle^+$$

$a_j \in \{S_1, S_2\}$, $\quad M := a_1 \cdots a_k \in GL_2(\mathbb{Z})$

$M\alpha = \alpha$

$\#\{\gamma \in \Gamma \mid \|\gamma\| < T\} \asymp T^{2\delta + o(1)}$ (Hensley, Lalley)

$\delta = 0.51\ldots = \dim_H$ (cont frac exp uses only 1 and 2)



LG-Conj: $\#\{\gamma \in \Gamma \mid \operatorname{tr}\gamma = t\} > t^{\eta + o(1)}$, $\eta = 2\delta - 1$

<u>Thm:</u> LG $\Rightarrow$ AC

<u>Quasi-random groups / super-approximation</u>

Bernstein - Kazhdan, Sarnak - Xue (wanted easy proof of Selberg 3/16 Thm) realized the idea

<u>Def:</u> G fin group. G is quasirandom group if

$\exists \alpha > 0 \ \forall \pi \in \hat{G}, \pi \neq \pi_0$: $\dim \pi > |G|^{\alpha}$

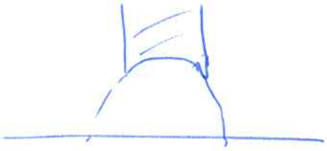
trivial repr

e.g., $G = PSL_2(\mathbb{F}_p)$

Frobenius $\Rightarrow$ $\dim \pi > \frac{p-1}{2} \asymp |G|^{1/3}$

$|G| \asymp p^3$ (exercise)

$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$

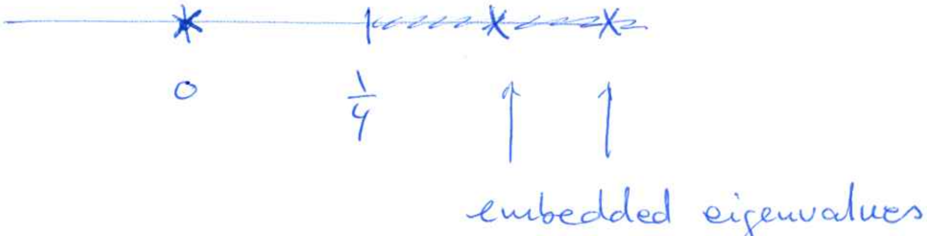


$$d\mu = \frac{dx\,dy}{y^2}$$

$$\Delta = y^2(\partial_x^2 + \partial_y^2)$$

$\Delta \curvearrowright L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$

spec Δ :

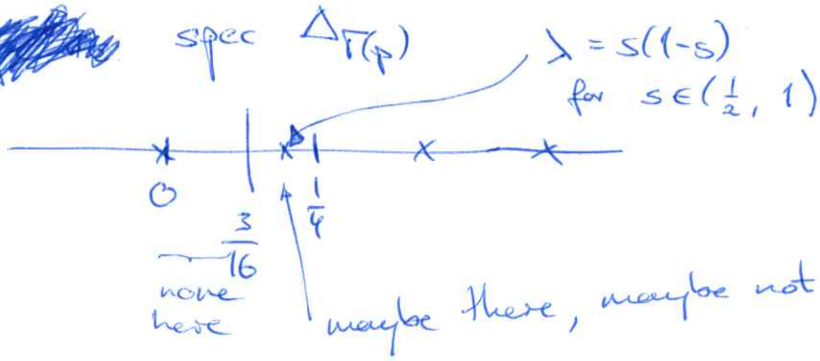


$\Gamma(p) = \{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I \mod p \}$, p prime

$\Gamma(p)\backslash\mathbb{H}$ $\qquad [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(p)] \simeq p^2$

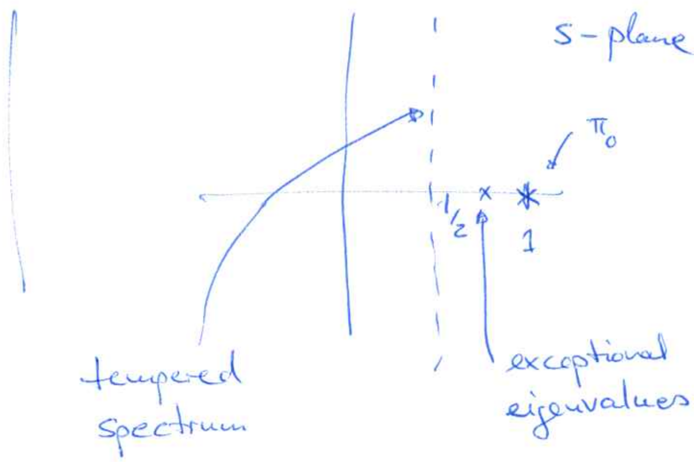
want to know spec $\Delta_{\Gamma(p)}$

spec $\Delta_{\Gamma(p)}$:



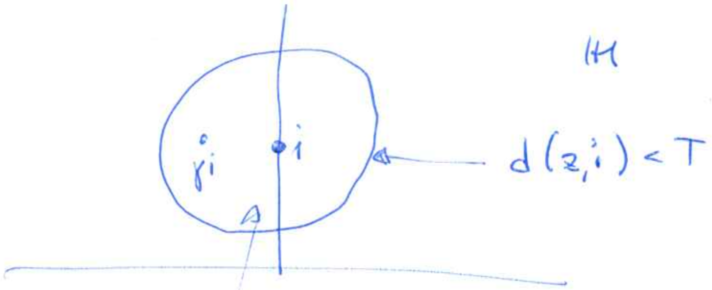
sketch of Selberg 3/16 :

$$\int_{\Gamma(p)\backslash\mathbb{H}} K_p(z,z)\,d\mu$$



$$T^{2s} m(\lambda) \leqslant \int_{\Gamma(p)\backslash \mathbb{H}} K_{p,T}(z,z)\,d\mu \ll \left| \Gamma(p)\backslash \Gamma \right| \Big(T + \sum_{\gamma \in \Gamma(p)} \mathbb{1}_{\{\|\gamma\| < T\}} \Big)$$

multiplicity of repr responsible for this exceptional eigenvalue λ



how many points in here?

sum counts:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad |a|, |b|, |c|, |d| < T, \quad a \equiv 1 \equiv d \ (p), \ b \equiv 0 \equiv c \ (p)$$

$$ad - bc = 1$$

Take 1: $\quad \frac{T}{p} \underbrace{+1}_{\text{doesn't forget this}}$ choices for a

choices for d: $\frac{T}{p} + 1$

now $ad - 1$ fixed

$bc = ad - 1 \Rightarrow b$ divisor of $ad - 1$

Fact: if $n < T$ then $\sigma_0(n) = \sum_{m|n} 1 \ll T^{\varepsilon}$

Total count: $N_p(T) \ll \left(\frac{T}{p} + 1\right)^2 T^{\varepsilon} \ll \frac{T^2}{p^2} + \frac{T}{p} + 1$

pretrace:

$$T^{2s} m(\lambda) \ll p^3 \left(T + \frac{T^2}{p^2}\right)$$

$$\Rightarrow \quad T^{2s} p^{\alpha} \ll p^3 \left(T + \frac{T^2}{p^2}\right)$$

quasirandom exponent

Choose T so that $T = \frac{T^2}{p^2} \Rightarrow T = p^2$

$$p^{4s} p^{\alpha} < p^3 (p^2)$$

$$\Rightarrow \quad 4s + \alpha \leq 5$$

here: $\alpha = 1$

$$\Rightarrow \quad 4s \leq 4 \Rightarrow s \leq 1 \qquad \text{tells nothing}$$

one needs something better in the argument

Sarnak - Xue: make count slightly better

go back to $ad - bc = 1$:

$$a = 1 + \alpha p \;, \quad c = \gamma p$$
$$b = \beta p \;, \quad d = 1 + \delta p$$

$$1 = (1 + \alpha p)(1 + \delta p) - \beta p \cdot \gamma p = 1 + (\alpha + \delta) p + (\alpha \delta - \beta \gamma) p^2$$

$$\Rightarrow \quad \alpha + \delta + (\alpha \delta - \beta \gamma) p = 0$$

$$\Rightarrow \quad \alpha + \delta \equiv 0 \mod p$$

$$\mathrm{Tr}\,\gamma = a+d = 1+\alpha p + 1 + \delta p = 2 + (\alpha+\delta)p$$

$$\equiv 2 \mod p^2$$

note!

new count $N_p(T)$:

choose ~~...~~ $\mathrm{Tr}\,\gamma$ first: # choices: $\frac{T}{p^2}+1$

choose a: $\frac{T}{p}+1$ $\Rightarrow$ know $d = \mathrm{tr} - a$

$$N_p(T) \ll \left(\frac{T}{p^2}+1\right)\left(\frac{T}{p}+1\right)T^{\varepsilon}$$

$$\ll \frac{T^2}{p^3} + \frac{T}{p} + 1$$

put count in pretrace formula:

$$T^{2s}m(\lambda) \ll p^3\left(T + \frac{T^2}{p^3}\right)$$

choose $T = p^3$

$$\Rightarrow \quad p^{6s}\underbrace{m(\lambda)}_{p^{\alpha}} < p^3(p^3)$$

$$\Rightarrow \quad 6s + \alpha < 6$$

$$\Rightarrow \quad s < \frac{6-\alpha}{6} = 1 - \frac{\alpha}{6}$$

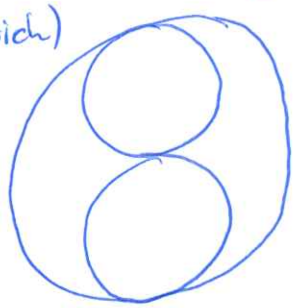
$\rightsquigarrow$ get $5/36$ bound.

Gamburd: used this trick for infinite co-area groups

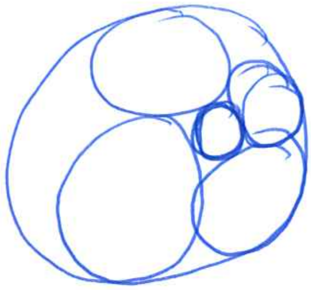
Crystallographic packings

(with Nakamura)
(with Kapovich)

given



can find $\underline{\underline{3}}$ new



<u>Def:</u> $\mathcal{P}$ sphere packing in $\hat{\mathbb{R}}^n$: collection of oriented spheres S^{n-1} with disjoint interiors, and interiors are dense in $\hat{\mathbb{R}}^n$.

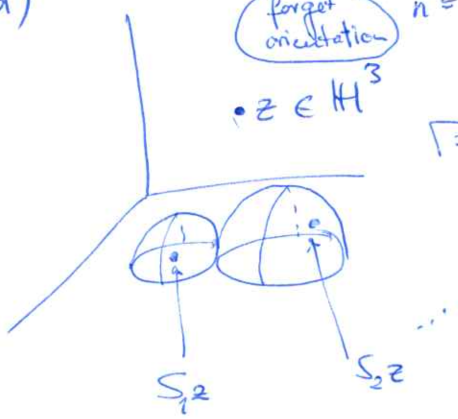
<u>note:</u> $\hat{\mathbb{R}}^n = \partial \mathbb{H}^{n+1}$

<u>Def:</u> $\mathcal{P}$ <u>integral</u> if all bends in $\mathbb{Z}$

<u>Def:</u> $\mathcal{P}$ is <u>crystallographic</u> if $\exists\ \Gamma < \mathrm{Isom}\, \mathbb{H}^{n+1}$ discrete reflective group s.th. $\overline{\mathcal{P}'} = \Lambda_\Gamma$ (limit set)
(finitely generated)

forget orientation

$n = 2$



$\Gamma = \langle S_1, -, S_4 \rangle \curvearrowright \mathbb{H}^3$

limit set is ACP

<u>Note:</u> Γ is thin.

Thm (K. – Nakamura)

P crystallographic $\Rightarrow$ $n < 1000$

(largest known: $n = 20$)

Structure Thm: If P crystallographic w/ symmetry group Γ, then $\exists$ $\tilde{\Gamma} < \mathrm{Isom}\, \mathbb{H}^{n+1}$ lattice: $\Gamma < \tilde{\Gamma}$

($\tilde{\Gamma}$ reflective)