On Hausdorff dimension of thin nonlinear solenoids

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joint work in progress with Feliks Przytycki (IMPAN) and Michał Rams (IMPAN)

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Hyperbolic set

Definition

Let $M$ be a $C^1$ Riemannian manifold, $U \subset M$ a non-empty open subset, $f : U \to f(U)$ a $C^1$ diffeomorphism. A compact $f$-invariant subset $\Lambda$ is **hyperbolic set** if there are constants $\lambda \in (0, 1)$, $C > 0$ and families of subspaces $E^{s/u}(x) \subset T_xM$, $x \in \Lambda$, s.t. for every $x \in \Lambda$:

- $T_xM = E^s(x) \oplus E^u(x)$;
- $\|D(f^n)|_{E^s(x)}\| \leq C\lambda^n$ for $n \in \mathbb{N}$;
- $\|D(f^{-n})|_{E^u(x)}\| \leq C\lambda^n$ for $n \in \mathbb{N}$;
- $Df(E^{s/u})(x) = E^{s/u}(f(x))$. 

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Let $M := S^1 \times \mathbb{D}$ be the solid torus, where $\mathbb{D} = \{ v \in \mathbb{R}^2 \mid |v| < 1 \}$ carries the product distance $d = d_1 \times d_2$ and suppose $f : M \to M$ such that

$$(x, y, z) \mapsto (\eta(x, y, z) \mod 2\pi, \lambda(x, y, z) + u(x), \nu(x, y, z) + v(x))$$

is a smooth embedding map, where $\lambda(x, 0, 0) = \nu(x, 0, 0) = 0$ and the component functions $\eta, \lambda, \nu : M \to \mathbb{R}$ satisfy the following assumptions:

1. $\eta'(x, y, z) := \frac{\partial}{\partial x} \eta(x, y, z) > 1$.
2. $\lambda'(x, y, z) := \frac{\partial}{\partial y} \lambda(x, y, z) < 1$.
3. $\nu'(x, y, z) := \frac{\partial}{\partial z} \nu(x, y, z) < \lambda'(x, y, z) < 1$.

$\Lambda := \cap_{n \in \mathbb{N}} f^n(M)$ is hyperbolic attractor. We define the natural projection $\pi : (x, y, z) \mapsto x$. For any set $D \subset M$, let $p \in D_x := (\pi|_D)^{-1}(x)$. Thus, $\Lambda_x := W^s_{D_x}(p) \cap \Lambda$. That is called Stable slice.
Solenoid: an example of a hyperbolic set

Let $x \mapsto 2x \mod 2\pi$ on $S^1$.

$\Lambda := \bigcap_{n \in \mathbb{N}} f^n(M)$ is hyperbolic attractor. $\Lambda_x$ is Stable slice.
**Conjecture.** The fractal dimension of a hyperbolic set is (at least generically or under mild hypotheses) the sum of those of its stable and unstable slices, where fractal can mean either Hausdorff or upper box dimension.
Previous results

1. Bothe (1993): If $f : M \to M$ is smooth embedding such that

$$(x, y, z) \mapsto (\eta(x) \mod 2\pi, \lambda_1(x) y + u(x), \lambda_2(x) z + v(x))$$

with $\eta'(x) = \frac{\partial}{\partial x} \eta(x) > 1$, $0 < \lambda_i < 1$ and $\sup \lambda_i < d^{-2}$ (d the mapping degree of $\eta$). Moreover, its unstable foliation satisfy transversality. Then, $\dim_H(\Lambda) = 1 + \dim_H(\Lambda_x)$. 

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2. Barriera, Pesin and Schmeling (1999): They established a dimension product structure of invariant measures in the course of proving the Eckmann Ruelle conjecture.
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2. Barter, Pesin and Schmeling (1999): They established a dimension product structure of invariant measures in the course of proving the Eckmann Ruelle conjecture.

3. Hasselblat and Schmeling (2004): If $f : M \to M$ is $C^2$ map such that

$$(x, y, z) \mapsto (2x \mod 2\pi, \lambda_1 y + u(x), \lambda_2 z + v(x))$$

with $0 < \lambda_2 < \lambda_1 < 1$ and $2\lambda_1 < 1$. Moreover, its unstable foliation satisfy transversality. Then, $\text{dim}_H(\Lambda) = 1 + \frac{\log 2}{\log \lambda_1}$. 

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Consider a $C^{1+\epsilon}$ map $f : M \rightarrow M$ as the example, and assume that $\eta'$ is constant as well as

1- $\sup \lambda'(p) < 1/d$ (d is degree) for $p \in \Lambda$,
2- The unstable lines of the $\pi(\Lambda)$ intersect each other transversal.

Then $\dim_H(\Lambda) = 1 + \dim_H(\Lambda_x)$ for every $x \in S^1$. 
Let $P = P_{f^{-1}}$ be topological pressure for the transformation $f^{-1}$ and let $\phi(p) = t \log \lambda(p)$ be a potential. Choose $t_0$ so that $P(t_0 \phi) = 0$.

Bowen (1975) shows that there is a unique equilibrium state $\mu_{t_0}$ for $t_0 \phi$. Moreover, $\mu_{t_0}$ has Gibbs properties.
Theorem B(M, Przytycki, Rams)

Assume that $\eta'$ is not constant. The previous theorem holds for regular points if instead of $\sup \lambda'(p) < 1/d$ we assume

$$
\chi_{\mu_{t_0}}(\lambda) := \int \log \lambda' \, d\mu_{t_0} < \chi_{\mu_{t_0}}(-\eta) := \int -\log \eta' \, d\mu_{t_0}.
$$
Danke!