On Hausdorff dimension of thin nonlinear solenoids

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Hyperbolic set

Definition

Let M be a C^1 Riemannian manifold, $U \subset M$ a non-empty open subset, $f : U \to f(U)$ a C^1 diffeomorphism.
A compact f-invariant subset Λ is **hyperbolic set** if there are constants $\lambda \in (0,1)$,$C > 0$ and families of subspaces $E^{s/u}(x) \subset T_xM$, $x \in \Lambda$, s.t. for every $x \in \Lambda$:

- $T_xM = E^s(x) \oplus E^u(x)$;
- $||D(f^n)|_{E^s(x)}|| \leq C\lambda^n$ for $n \in \mathbb{N}$;
- $||D(f^{-n})|_{E^u(x)}|| \leq C\lambda^n$ for $n \in \mathbb{N}$;
- $Df(E^{s/u})(x) = E^{s/u}(f(x))$.

Solenoid: an example of a hyperbolic set

Let $M := S^1 \times \mathbb{D}$ be the solid torus, where $\mathbb{D} = \{v \in \mathbb{R}^2 || v| < 1\}$ carries the product distance $d = d_1 \times d_2$ and suppose $f : M \to M$ such that

$$(x, y, z) \mapsto (\eta(x, y, z) \mod 2\pi, \lambda(x, y, z) + u(x), \nu(x, y, z) + v(x))$$

is a smooth embedding map, where $\lambda(x, 0, 0) = \nu(x, 0, 0) = 0$ and the component functions $\eta, \lambda, \nu : M \to \mathbb{R}$ satisfy the following assumptions :

1- $\eta'(x, y, z) := \frac{\partial}{\partial x}\eta(x, y, z) > 1.$

2- $\lambda'(x, y, z) := \frac{\partial}{\partial y}\lambda(x, y, z) < 1.$

3- $\nu'(x, y, z) := \frac{\partial}{\partial z}\nu(x, y, z) < \lambda'(x, y, z) < 1.$

$\Lambda := \cap_{n \in \mathbb{N}} f^n(M)$ is hyperbolic attractor. We define the natural projection $\pi : (x, y, z) \mapsto x$. For any set $\mathcal{D} \subset M$, let $p \in \mathcal{D}_x := (\pi_{|\mathcal{D}})^{-1}(x)$. Thus, $\Lambda_x := W^s_{\mathcal{D}_x}(p) \cap \Lambda$. That is called *Stable slice*.

Solenoid: an example of a hyperbolic set

Let $x \mapsto 2x \; mod 2\pi$ on S^1.

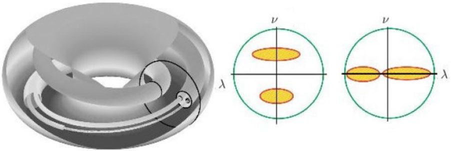

Figure: Λ_x

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Eckmann Ruelle conjecture

Conjecture. The fractal dimension of a hyperbolic set is (at least generically or under mild hypotheses) the sum of those of its stable and unstable slices, where fractal can mean either Hausdorff or upper box dimension.

Previous results

1. Bothe (1993) : If $f : M \to M$ is smooth embedding such that

$$(x, y, z) \mapsto (\eta(x) \mod 2\pi, \lambda_1(x)y + u(x), \lambda_2(x)z + v(x))$$

with $\eta^{'}(x) = \frac{\partial}{\partial x}\eta(x) > 1$, $0 < \lambda_i < 1$ and $sup\lambda_i < d^{-2}$(d the mapping degree of η). Moreover, its unstable foliation satisfy transversality. Then, $dim_H(\Lambda) = 1 + \dim_H(\Lambda_x)$.

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3. Hasselblat and Schmeling (2004): If $f : M \to M$ is C^2 map such that

$$(x, y, z) \mapsto (2x \mod 2\pi, \lambda_1 y + u(x), \lambda_2 z + v(x))$$

with $0 < \lambda_2 < \lambda_1 < 1$ and $2\lambda_1 < 1$. Moreover, its unstable foliation satisfy transversality . Then, $dim_H(\Lambda) = 1 + \frac{\log 2}{\log \lambda_1}$.

Theorem A(M, Przytycki, Rams)

Theorem

Consider a $C^{1+\epsilon}$ map $f : M \to M$ as the example, and assume that $\eta^{'}$ is constant as well as

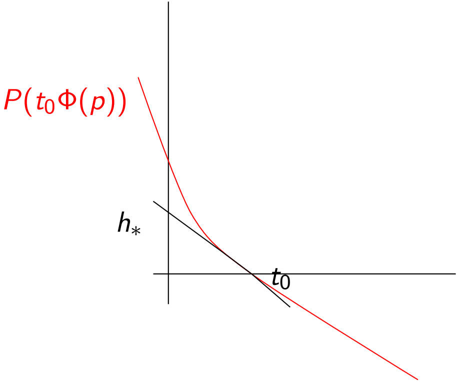
1- $sup\lambda^{'}(p) < 1/d($ d is degree) for $p \in \Lambda$,

2- The unstable lines of the $\pi(\Lambda)$ intersect each other transversal.

Then $\dim_H(\Lambda) = 1 + \dim_H(\Lambda_x)$ for every $x \in S^1$.

Geometric measure

Let $P = P_{f^{-1}}$ be topological pressure for the transformation f^{-1} and let $\phi(p) = t \log \lambda(p)$ be a potential. Choose t_0 so that $P(t_0\phi) = 0$. Bowen(1975) shows that there is a unique equilibrium state μ_{t_0} for $t_0\phi$. Moreover, μ_{t_0} has Gibbs properties.



Theorem B(M, Przytycki, Rams)

Theorem

Assume that $\eta^{'}$ is not constant. The previous theorem holds for regular points if instead of $sup\lambda^{'}(p) < 1/d$ we assume $\chi_{\mu_{t_0}}(\lambda) := \int \log \lambda^{'} d\mu_{t_0} < \chi_{\mu_{t_0}}(-\eta) := \int -\log \eta^{'} d\mu_{t_0}$.

Thanks

Danke!