Optimal balance for geophysical flows and spontaneous wave emission

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The rotating shallow water model

Primitive variables:

Velocity and height fields: $(\boldsymbol{u}, h)$.

Non-dimensional model:

$$\varepsilon\,(\partial_t \boldsymbol{u} \;+\; \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}) + \boldsymbol{u}^{\perp} + \boldsymbol{\nabla} h = 0\,,$$
$$\partial_t h + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) + h_0\,\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,,$$

$$h_0 = \frac{H_0}{H}\,,\quad T = \frac{L}{U}\,,\quad \varepsilon = \frac{U}{fL}\,.$$

Geostrophic balance:

$\varepsilon \ll 1\,,\;\; \boldsymbol{u}_G = \boldsymbol{\nabla}^{\perp} h\,.$

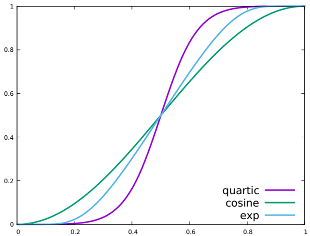
Semi-geostrophic scaling:

$h = O(1)\,,\; t = O(1/\varepsilon)\,.$

Optimal balance

Underlying idea:

- Build a "slow manifold", $(\delta, \gamma) = F(q)$,
- slow deformation to the linear shallow water equations,
- ramp function $\rho : [0,1] \to [0,1]$.



Optimal balance model: The boundary value problem

$$\varepsilon(\partial_t \boldsymbol{u} + \rho\,(t/T)\,\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}) + \boldsymbol{u}^\perp + \boldsymbol{\nabla} h = 0\,,$$
$$\partial_t h + \rho\,(t/T)\,\boldsymbol{\nabla} \cdot (h\boldsymbol{u}) + h_0 \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,,$$

with linear-end and nonlinear-end boundary conditions,

$$\mathbb{P}_G(\hat{\boldsymbol{u}}, \hat{h}) = 0\,, \qquad q(T) = q\,.$$

Gottwald, Mohammad, Oliver (2017).

Linear-end boundary condition

Task: Build projection matrices to decompose $(\boldsymbol{u}, h)$.

Linear model: At $t = 0$,

$$\varepsilon \partial_t \boldsymbol{u} + \boldsymbol{u}^{\perp} + \boldsymbol{\nabla} h = 0,$$
$$\partial_t h + h_0 \, \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,,$$

- for spectral representation $\hat{\boldsymbol{z}} = (\hat{u}, \hat{v}, \hat{h})$:

$$\frac{\partial}{\partial t} \hat{\boldsymbol{z}} = i A \hat{\boldsymbol{z}} \,, \quad A = \begin{bmatrix} 0 & -i/\varepsilon & -k/\varepsilon \\ i/\varepsilon & 0 & -l/\varepsilon \\ -h_0 k & -h_0 l & 0 \end{bmatrix} \,,$$

- Rossby (geostrophic) mode: $w_0 = 0$, and
  gravity-wave (ageostrophic) modes: $w_{1,2}^2 = (\varepsilon h_0 (k + l)^2 + 1)/\varepsilon^2$,
- define projections $\mathbb{P}_R$ and $\mathbb{P}_G = (I - \mathbb{P}_R)$:

$$\hat{z} = \mathbb{P}_G \hat{z} + \mathbb{P}_R \hat{z} \,.$$

Primitive and geostrophic-ageostrophic variables

Task: $(\boldsymbol{u},\ h) \ \Rightarrow\ (q,\ \delta,\ \gamma)$

Geostrophic variable:

$$q = \frac{\varepsilon \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{u} + 1}{h_0 + h}$$

Ageostrophic variables:

$$\delta = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\gamma = \boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{u} - \Delta h$$

Task: $(q,\ \delta,\ \gamma) \ \Rightarrow\ (\boldsymbol{u},\ h)$

PV inversion equations:

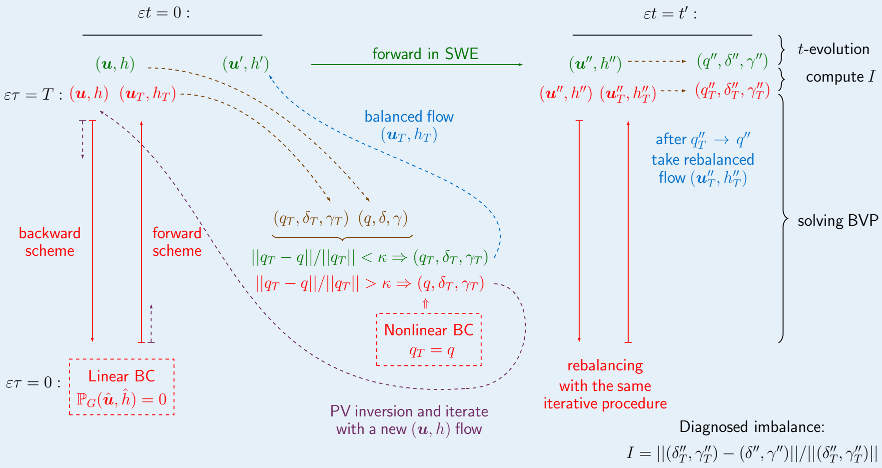
$$(-q + \varepsilon\Delta)h = -\varepsilon\gamma + q h_0 - 1\,,$$

$$\boldsymbol{u} = \boldsymbol{\nabla}^{\perp}\psi + \boldsymbol{\nabla}\phi + \bar{\boldsymbol{u}},$$

$$\Delta\psi = \zeta, \qquad \Delta\phi = \delta.$$

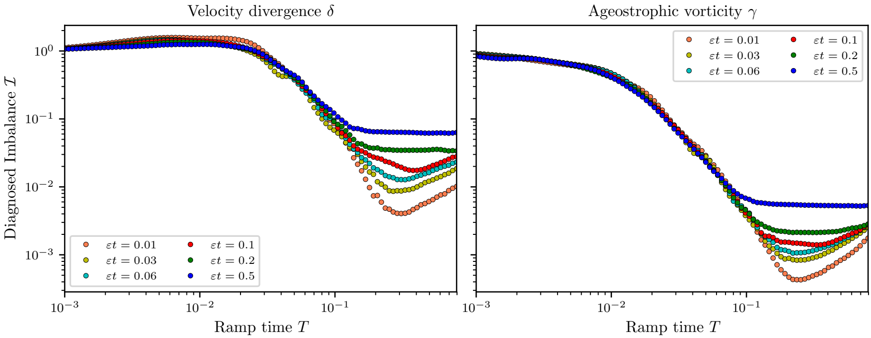
Optimal balance algorithm

Schematic of optimal balance to the rotating SWE

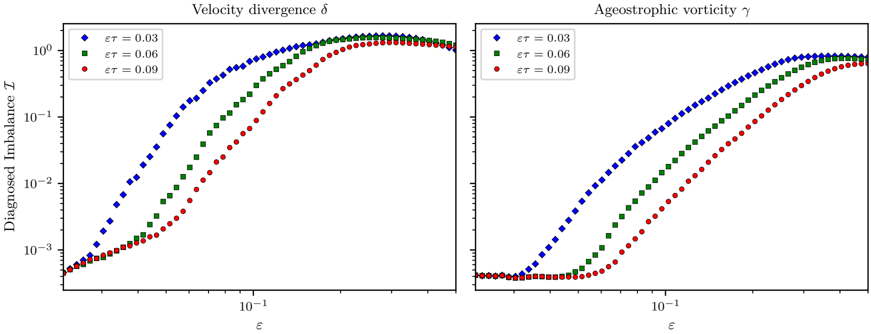


Diagnosed imbalances $I(T)$ for $\varepsilon = 0.1$

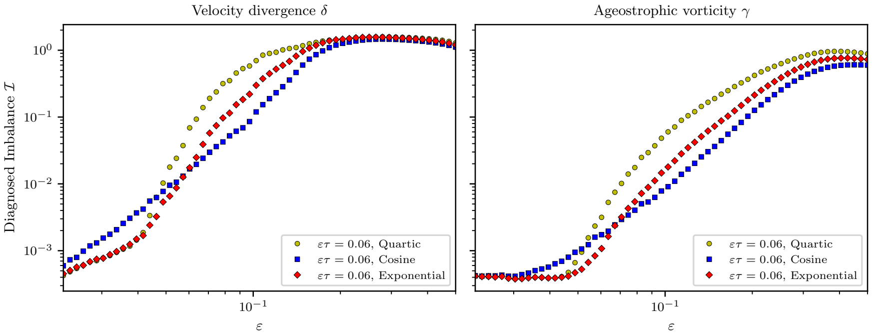
$\varepsilon = 2^{-m/2}$, $m = 2, \cdots, 11$



Diagnosed imbalances $I(\varepsilon)$, $\varepsilon t = 0.03$



$I(\varepsilon)$ for different ramp functions ρ, $\varepsilon t = 0.06$



Conclusion

- Longer ramp time T up to optimal value provides smaller I .
- The method works with balance state q and oblique projection at optimal computational effort.
- Linear BCs (oblique and orthogonal projection) have an impact on the process but not on the solution that the algorithm converges.
- Cosine ramp function excites smaller imbalances for middle-range ε values.