Homoclinic orbits in the Circular Restricted Four Body Problem

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Goal:

Develop a numerical scheme to find and validate the existence of homoclinic orbits in the circular restricted four body problem at bifurcation points.
Circular restricted four body problem
Laws of motion

\[
\begin{align*}
\ddot{x} &= 2\dot{y} + \partial_x \Omega(x, y) \\
\ddot{y} &= -2\dot{x} + \partial_y \Omega(x, y)
\end{align*}
\]

with \( \Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=1}^{3} \frac{m_i}{\sqrt{(x-x_i)^2+(y-y_i)^2}}. \)
Circular restricted four body problem

Dynamics in CRFBP

Laws of motion

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with \( \Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=1}^{3} \frac{m_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2}}. \)

Position primary bodies

\[ x_1 = -M \]
\[ x_2 = \frac{2m_2 + m_3 - 2M^2}{2M} \]
\[ x_3 = \frac{m_2 + 2m_3 - 2M^2}{2M} \]
\[ y_1 = 0 \]
\[ y_2 = -\frac{\sqrt{3}m_3}{2M} \]
\[ y_3 = \frac{\sqrt{3}m_2}{2M} \]

with constant \( M = \sqrt{m_2^2 + m_2m_3 + m_3^3}. \)
Circular restricted four body problem

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Dynamics in CRFBP

Laws of motion

\[ \ddot{x} = 2\dot{y} + \partial_x \Omega(x, y) \]
\[ \ddot{y} = -2\dot{x} + \partial_y \Omega(x, y) \]

with \( \Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{i=1}^{3} \frac{m_i}{\sqrt{(x-x_i)^2+(y-y_i)^2}} \).

Assumptions on masses

\[ \sum_{i=1}^{3} m_i = 1 \quad \text{and} \quad 0 < m_3 \leq m_2 \leq m_1 < 1. \]

Energy conservation

\[ E := \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x, y). \]
Lagrangian points and bifurcations

10 Lagrangian points for $m_1 = 0.38$ and $m_2 = 0.35$. 
Lagrangian points and bifurcations

10 Lagrangian points for $m_1 = 0.40$ and $m_2 = 0.35$. 

The three primary bodies
The Lagrangian points
10 Lagrangian points for $m_1 = 0.42$ and $m_2 = 0.35$. 
Lagrangian points and bifurcations

9 Lagrangian points for \( m_1 \approx 0.4247 \) and \( m_2 = 0.35 \).
Lagrangian points and bifurcations

8 Lagrangian points for $m_1 = 0.46$ and $m_2 = 0.35$. 
The bifurcation curve in the \((m_1, m_2)\)-plane.
Hamiltonian saddle node bifurcation

Bifurcation diagram of a two dimensional Hamiltonian saddle node bifurcation.
The non-linear stable and unstable manifolds with their dynamics
Numerical approach to homoclinic orbits

1. Find a point on the bifurcation curve.

2. Compute the center, center-stable and center-unstable manifolds and conjugate dynamics (up to high order).

3. Compute the appropriate 2 dimensional manifolds $W_u$ and $W_s$ (up to high order).

4. Find a global connection between $W_u$ and $W_s$.

5. Validate the solution.
Existence of invariant manifolds

**Theorem (Parameterization method for center manifolds)**

There exists a $K_c : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ and $R_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ s.t.

\[
\begin{align*}
F \circ K_c &= DK_c \cdot R_c \text{ locally} \\
K_c \text{ is tangent to the center subspace} \\
R_c \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} y \\ 0 \end{pmatrix} + h.o.t.
\end{align*}
\]
Numerical results

Numerical computations

• Numerically compute 15\textsuperscript{th} order Taylor polynomials for the manifolds.

• Restrict the non-linear (un)stable manifold such that the error of the conjugacy equation is of the order $10^{-16}$.

• Size of the non-linear (un)stable manifold is of the order $10^{-2}$.

• Use an ODE-solver to find the homoclinic orbit.
The invariant manifold

The $\dot{y}$-coordinate of the unstable manifold and its error.
The flow on the boundary of the stable (Blue) and unstable (Red) manifold

The numerical homoclinic orbit.
The homoclinic orbit

The flow on the boundary of the stable (Blue) and unstable (Red) manifold

The numerical homoclinic orbit.
Thanks for your attention