On Non-Linear Pseudo Anosov map and parabolic flows

P. Giulietti

Scuola Normale Superiore
Centro di Ricerca Matematica “Ennio de Giorgi”
Pisa - Italy

7th Bremen Summer School and Symposium
Bremen - August 5, 2019

A cohomological equation prologue

- Let $V \in C^r(M)$ vector field. Let ϕ_t be the flow defined by the differential equation

$$\frac{d}{dt}(\phi_t(x)) = V(\phi_t(x)).$$

Given $g : M \to \mathbb{R}$, we would like to study the growth of

$$H_{x,t}(g) := \int_0^t ds\, g \circ \phi_s(x).$$

- If there exists an $h : M \to \mathbb{R}$ such that

$$h \circ \phi_t - h = \int_0^t ds\, g \circ \phi_s \text{ then } g = \langle V, \nabla h \rangle.$$

- It can be used to classify flows w.r.t. reparametrizations.
- Suppose ϕ_t is uniquely ergodic. If $\mu(g) \neq 0$ then $H_{x,t}(g) \sim t\mu(g)$. If $\mu(g) = 0$?
- The growth of $H_{x,t}(g)$ is not a topological invariant.

A Special choice of V

- Let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism. Let $V(x) = E^s_F(x)$ i.e. its stable normalized vector field.
- $\{\phi_t(x)\}_{t\in I}$ is a piece of the stable manifold for F, thus

$$F^n(\phi_t(x)) = \phi_{\tau_n(x,t)}(F^n(x))$$

Key fact

$$H_{x,t}(g) = \int_0^t ds\, g \circ F^{-n} \circ F^n \circ \phi_s(x) = H_{F^n(x),\tau_n(x,t)}(\mathcal{L}^n_F g).$$

- $\mathcal{L}_F$ is a transfer operator, $\tau_n(x,t)$ is the rescaled time.
- $\mathcal{L}_F$ is quasi-compact on $\mathcal{B}^{p,q}$: an anisotropic Banach space of distributions which are p regular in the unstable direction and are integrated q times along the stable one.

"Prototypical" growth

Theorem (G. - Liverani '14)

Given $p+q$ *large,* $\exists\, N_1$ obstructions $\{O_i\}_{i=1}^{N_1}$*, sets* $\mathbb{V}_k, k \le N_1$ *s.t.*

$$\mathbb{V}_k := \{g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{C}) \ : \ O_j(g) = 0 \ \forall j < k \,;\, O_k(g) \neq 0\}.$$

Then $\exists C > 0$ *such that, for all* $x \in \mathbb{T}^2$ *and* $g \in \mathbb{V}_k$*, we have*

$$|H_{x,t}(g)| \le C\, t^{\alpha_k} (\ln t)^{b_k} \|g\|_{\mathcal{C}^r}$$

s.t. α_k, b_k *are given by the spectral data of* $\mathcal{L}_F$*.*

Remark: In the toy model of a Torus, given an Anosov diffeomorphism, there are no such deviations from linear growth. (Forni '19, Baladi '19)

The vertical flow of a "Linear" Pseudo-Anosov Map

Setup

- X compact connected surface, genus g, Σ singularities.
- F pseudo-Anosov map, $\lambda > 1$ constant expansion, preserving orientation.
- ϕ_t be the vertical flow, $f \in \mathrm{Obs}(M \setminus \Sigma)$.

Theorem (Faure-Gouëzel-Lanneau '18)

Assuming $f \in C_h^2$ *and* $\omega(f) = 0, \forall \omega \in O_{\lambda,0}(\mathcal{L}_F)$ *then there exists* $F \in C^0$ *such that*

$$\int_0^\tau f(\phi_t x)dt = F(x) - F(\phi_\tau(x))$$

The vertical flow of a "Linear" Pseudo-Anosov Map

Setup

- X compact connected surface, genus g, Σ singularities.
- F pseudo-Anosov map, $\lambda > 1$ constant expansion, preserving orientation.
- ϕ_t be the vertical flow, $f \in \mathrm{Obs}(M \setminus \Sigma)$.

Theorem (Faure-Gouëzel-Lanneau '18)

Assuming $f \in C_h^{k+2}$ *and* $\omega(f) = 0, \forall \omega \in O_{\lambda,k,}(\mathcal{L}_T)$ *then there exists* $F \in C_h^k$ *along the horizontal direction and extends continuously to* M *such that*

$$\int_0^\tau f(\phi_t x)dt = F(x) - F(\phi_\tau(x))$$

Linear functionals $O_{\lambda,k}$: combination of transfer operator eigendistributions and eigenvalues of F^* on H_1.

The flow of a Perturbed Pseudo-Anosov map

Setup

- X compact connected surface, genus g, Σ singularities.
- F "perturbation" of a pseudo-Anosov map.
- $\phi_t(x)$ be the acceleration flow related to $E_s(x), x \in M \setminus \Sigma$.
  $f \in \mathrm{Obs}(M \setminus \Sigma)$.

Theorem (Artigiani, G. '19)

Assuming f is sufficiently regular and $\omega(f) = 0, \forall \omega \in O_{\lambda,0}(\mathcal{L}_F)$ then there exists $F \in C^0$ such that

$$\int_0^\tau f(\phi_t x)dt = F(x) - F(\phi_\tau(x)).$$

The flow of a Perturbed Pseudo-Anosov map

Setup

- X compact connected surface, genus g, Σ singularities.
- F "perturbation" of a pseudo-Anosov map.
- $\phi_t(x)$ be the acceleration flow related to $E_s(x), x \in M \setminus \Sigma$.
  $f \in \mathrm{Obs}(M \setminus \Sigma)$.

Theorem (Artigiani, G. '19)

Assuming f is sufficiently regular and $\omega(f) = 0, \forall \omega \in O_{\lambda,Lip}(\mathcal{L}_F)$ *then there exists* F *Lipschitz along the contracting leaf at* x *such that*

$$\int_0^\tau f(\phi_t x)dt = F(x) - F(\phi_\tau(x)).$$

The Vertical Flow On Generic Surfaces

Artigiani, G. (WIP)

- If one tries to study the vertical flow on a generic surface unluckily the Renormalization equation does not hold (as the surface is not a fixed point for the Teichmüller flow).
- The integral of an observable along the vertical flow can be decomposed, approximately, by bits living on regular surfaces.
- The Pseudo-Anosov maps which renormalize each bit are all linear and close to each other. They can be considered perturbations of each other, "one" functional space $\mathcal{B}^{p,q}$ is sufficient.