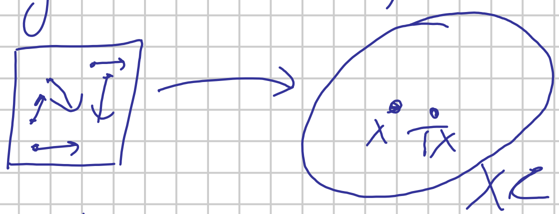
Ergodic Theorems

Bremen Summer School
5.–9. 8.19

1. Classical Ergodic Theorems

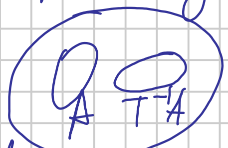
Origin: ~ 1880 Ludwig Boltzmann, statistical mechanics

Model: ideal gas
k particles



$X \subset \mathbb{R}^{6k}$ state space

$T: X \to X$
Vol-preserving



$Vol(T^{-1}A) = Vol(A)$

Boltzmann's ergodic hypothesis:

"time mean = space mean"

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1\!\!1_A (T^n x) = \int 1\!\!1_A \, d\mu$$

rescaled Vol

$\boxed{\text{Def.}}$ 1) A measure-pres. dynamical system (MDS) is (X, μ, T), where

- (X, μ) prob. space
- $T: X \to X$ μ-pres., i.e., $\mu(T^{-1}(A)) = \mu(A)$ $\forall A$ meas.

2) (X, μ, T) is **ergodic** if $\forall A \subset X$ meas.

$$A \ \underbrace{T\text{-inv.}}_{T^{-1}(A) \subset A \text{ up to a null set}} \Rightarrow \mu(A) \in \{0,1\}.$$

$\boxed{\text{Ex}}$ 1) Finite systems

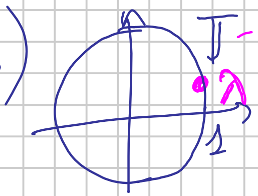
2) Bernoulli shifts

$b \in \mathbb{N}$ $\quad X = \{0, \dots, b-1\}^{\mathbb{N}}$, $T \leftarrow$ (left shift)

μ product measure corresp. to (any) prob. measure p on $\{0, \dots, b-1\}$

$$\mu(\underbrace{X \times \dots \times \{a_1\} \times X \times \dots \times \{a_j\} \times \dots}_{\text{cylinder set}}) := p(a_1) \cdot \ldots \cdot p(a_j)$$

$(X, \mu, \leftarrow)$ ergodic (even mixing: $\mu(T^{-n}A \cap B) \to \mu(A)\mu(B)$)

3) 

$\mathbb{T}$-unit circle

$$(\mathbb{T}, \underbrace{\text{Leb}}_{\text{rescaled}}, \underbrace{\lambda}_{\text{rotation by } \lambda}) \qquad Tz := \lambda \cdot z$$

erg. $\overset{\text{why?}}{\Longleftrightarrow}$ λ irrat. (i.e., $\lambda^n \neq 1 \ \forall n \neq 0$)

More generally: rotations on compact groups w.r.t. $\underbrace{\text{Haar measure}}_{\text{unique rot-inv. measure}}$

4) (X, T) topological dynam. system (TDS)

X: comp. top. space; $T: X \to X$ cont.

Krylov-Bogolyubov: $\exists$ T-inv. prob. meas. μ on X $\rightsquigarrow (X, \mu, T)$

idea: Cons. $\mu_n = \frac{\delta_a + \ldots + \delta_{T^n a}}{n+1}$ „asympt. T-inv."

Take any weak*-limit point „[redacted]"

if μ is unique, (X, T) is called uniquely ergodic.

(Ex.: $(\mathbb{T}, \lambda)_{\text{irrat}}$ - why?)

Linearisation:

$A \rightsquigarrow \mathbb{1}_A$

$(X, \mu) \rightsquigarrow L^p(X, \mu)$, $p \in [1, \infty)$

transf. $T \rightsquigarrow$ Koopman operator $T: L^p \to L^p$

$(Tf)(x) := f(Tx)$

Oper. T is:

- positive $(f \geq 0 \Rightarrow Tf \geq 0)$, multipl., $T\bar{f} = \overline{Tf}$

- isometric : $\|Tf\| = \|f\|$ $\forall f$ (exerc.)
- invert. if the transf. T invert.
- (X, μ, T) erg. $\iff$ $\mathrm{Fix}\,T = \langle \mathbb{1} \rangle$ (exerc.)

$\boxed{\text{Thm}}$ (Mean Ergodic Thm, von Neumann '33) $f : Tf = f$

$\forall$ Hilbert H, $T \in \mathcal{L}(H)$ isometr., $f \in H$

$$\frac{1}{N} \sum_{n=1}^{N} T^n f \longrightarrow P_{\mathrm{Fix}\,T} f$$

orthog. proj.

Sketch of proof:

Step 1 $H = \mathrm{Fix}\,T \oplus^{\perp} \overline{Rg(I-T)}$ (von Neumann decomposition)

Step 2 $f \in \mathrm{Fix}\,T$ – clear

$f \in Rg(I-T)$: $f = g - Tg$: $\frac{1}{N} \sum_{n=1}^{N} (T^n g - T^{n+1} g) = \frac{Tg - T^{N+1} g}{N+1} \to 0$ $\|\cdot\| \le 2\|g\|$

+ appr. arg.

" $\blacksquare$ "

$\boxed{\text{Thm}}$ (Pointwise Erg. Thm, Birkhoff '33)

$\forall$ MDS (X, μ, T) $\forall f \in L^1(X, \mu)$

$\frac{1}{N}\sum_{n=1}^{N} T^{n} f$

conv. a.e. (to $\mathbb{E}(f \mid \underbrace{\Sigma_{inv}}_{\{A \ T\text{-inv}\}})$, an equals $\int f d\mu \cdot \mathbb{1}$ if T erg.)

Rem. $f = \mathbb{1}_A$: $\frac{1}{N}\sum_{n=1}^{N} (T^{n} \mathbb{1}_A)(x) \to \int \mathbb{1}_A d\mu$ for a.e. x if T erg.

– Boltzmann's erg. hypoth. (only a.e., erg. systems)

Rem. 1) Birkhoff + erg. of Bernoulli shift imply (exerc) that a.e. (Lebesgue) $x \in [0,1]$ is normal, i.e., $x = 0, a_1 a_2 a_3 \dots$, $a_i \in \{0,\dots,9\}$ s.t.

$\forall d \quad \forall w_1 \cdots w_d \in \{0,\dots,9\}^d$

$$\frac{\#\{n \in \{1,\dots,N\} : a_n = w_1, \dots, a_{n+d-1} = w_d\}}{N} \longrightarrow \left(\frac{1}{10}\right)^d$$

(even in $\forall$ base).

Ex: $0,123\dots 9101112\dots$ – normal base 10

$\sqrt{2}$, $\log 2$, π, e, ...

2) Birkhoff for uniquely erg. systems

$\forall (X,T)$ uniq. erg. $\quad \forall f \in C(X)$

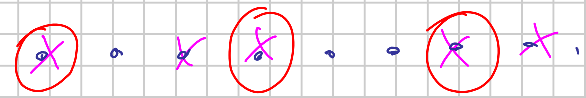
$$\frac{1}{N}\sum_{n=1}^{N} (T^n f)(x) \longrightarrow \int f \, d\mu \qquad \forall x \text{ (even unif. in } x\text{!)}$$

Exerc.

2. Multiple Ergodic Thms

Motivation: arithm. progressions in large sets

$C \subset \mathbb{N}$ satisf. (AP) if



$$\forall \underset{\text{length}}{b} \; \exists \underset{\text{beginning}}{a} \; \exists \underset{\text{step}}{n} : a, a+n, \dots, a+bn \in C$$

$\boxed{\text{Thm}}$ (Szemerédi '75) $\forall C$ with $\bar{d}(C) := \overline{\lim_{N\to\infty}} \frac{\#(C \cap \{1 \dots N\})}{N} > 0$ satisf. (AP).

$\boxed{\text{Thm}}$ (Furstenberg '77)

i) (Multiple recurrence) $\forall (X, \mu, T)$ $\forall f \in L^\infty$, $f > 0$ $\forall k$

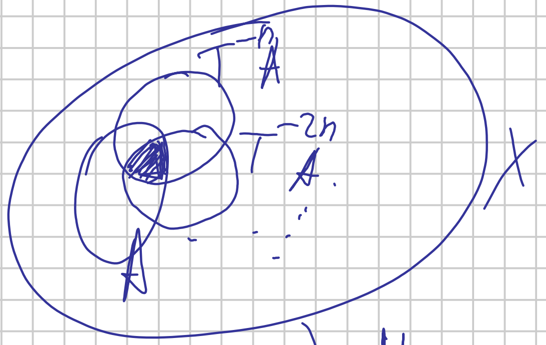
$f \geq 0, f \neq 0$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \int f \cdot T^n f \cdot \dots \cdot T^{kn} f > 0$$

2) (Correspondence Principle) Multipl. recurr. $\Longrightarrow$ Szemerédi

Rem. 1) Multiple recurr. for $f = \mathbb{1}_A$, $\mu(A) > 0$,

$$\mu\left(A \cap T^{-n}A \cap \dots \cap T^{-kn}A\right) > 0 \quad \text{for many } n$$



3) Many generalisations: Szemerédi in $\mathbb{Z}^d$, polynom. progressions $(a, a+n^2, \dots)$

4) Green-Tao '2004: $\mathbb{P}$ sat. (AP)

Ex: 7, 37, 67, 97, 127, 157, ~~187~~

longest known arithm. progr. in $\mathbb{P}$ length 26

5) Host-Kra '2002: multiple convergence

$$\frac{1}{N}\sum_{n=1}^{N} T^{n}f_1 \cdot T^{2n}f_2 \cdot \dots \cdot T^{kn}f_k$$

conv. in L^2 $\forall k$ $\forall f_1, \dots, f_k \in L^\infty$ ($k=2$ Furstenberg ~'80)

Idea (correspondence principle) $\quad C \subset \mathbb{N}$ with $\bar{d}(C) > 0$, $k \in \mathbb{N}$.

$C \rightsquigarrow \mathbb{1}_C \in \ell^\infty(\mathbb{Z})$, $\quad T : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ $\leftarrow$

$X := \overline{\{T^k \mathbb{1}_C,\ k \in \mathbb{Z}\}}^{\text{weak}^*}$

- $\forall x \in X$ is 0-1-seq.
- X is weak*-comp. (Banach-Alaoglu)

$A := \{(t_j) \in X : t_0 = 1\}$ - cylinder

- A is open
- $\boxed{T^m \mathbb{1}_C \in A \iff m \in C}$ (*)

C sat. $(AP)_k$ $\iff \exists a, n : \underbrace{a, a+n, \dots, a+kn \in C}$

by (*) $T^a \mathbb{1}_C, \dots, T^{a+kn} \mathbb{1}_C \in A$

$\iff \exists n\ \exists a\ \underbrace{T^a \mathbb{1}_C}_{\text{are dense}} \in \underbrace{A \cap T^{-n} A \cap \dots \cap T^{-kn} A}_{\text{open}}$

$\Longleftrightarrow \exists n: \quad A \cap T^{-n}A \cap \dots \cap T^{-kn}A \neq \emptyset$

$\overset{?}{\Longleftarrow} \quad \exists \mu \quad \exists n: \; \mu(\text{—\textquotedbl—}) > 0.$

<u>Construction of μ</u>: $\quad \exists N_k \nearrow \infty : \quad \underbrace{\frac{\#(C \cap \{1, \dots, N_k\})}{N_k}}_{(**)} \longrightarrow \bar{d}(C) > 0.$

$$(**) = \frac{1}{N_k} \sum_{n=1}^{N_k} \mathbb{1}_C(n) \underset{(*)}{=} \frac{1}{N_k} \sum_{n=1}^{N_k} \mathbb{1}_A(T^n \mathbb{1}_C)$$

$$= \underbrace{\frac{1}{N_k} \sum_{n=1}^{N_k} \delta_{T^n \mathbb{1}_C}}_{=: \mu_k \quad \text{asympt. inv.}} (A)$$

Take μ be any weak*-limit point:

$$\left.\begin{array}{l} \cdot \; \mu \text{ prob. m. on } X \\ \cdot \; \mu(A) = \bar{d}(C) > 0 \\ \cdot \; \mu \;\; T\text{-inv} \end{array}\right\} \text{exerc.}$$

Mult. recurr. $\Longrightarrow \mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0$ for many n $\blacksquare$

<u>Idea</u> (mult. conv. for erg. systems, $k=2$)

<u>Step 1</u> Decomp. $L^2 = \overline{\text{lin}}\{\text{EF of } T\} \oplus^{\perp} \{\text{w. mixing fcts}\}$

eigenfct

f: $\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle| \longrightarrow 0$

<u>Step 2</u> f, g EF $\Longrightarrow$ $\frac{1}{N}\sum_{n=1}^{N} \underbrace{\lambda_1^n \cdot \lambda_2^{2n}}_{(\lambda_1\lambda_2^2)^n} fg$ conv.

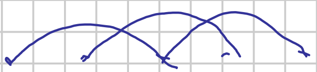
f or g w. mixing $\longrightarrow$ conv. to 0 by a trick

„van der Corput ▨"

<u>Van der Corput:</u>

$(u_n) \subset H$ bdd, H Hilbert,

$$\gamma_k := \overline{\lim_{N\to\infty}} \left| \frac{1}{N}\sum_{n=1}^{N} \langle u_{n+k}, u_n\rangle \right|$$



if $\frac{1}{N}\sum_{1}^{N} \gamma_k \longrightarrow 0$, then $\frac{1}{N}\sum_{1}^{N} u_n \longrightarrow 0$.

exerc.: γ_k for $u_n := T^n f \cdot T^{2n} g$

3. Weighted Ergodic Thms

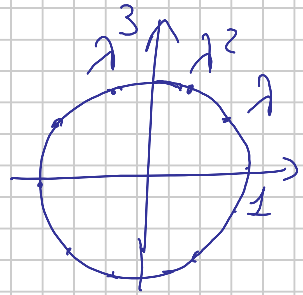
Question/goal: Find good (mostly: bdd) weights $(a_n) \subset \mathbb{C}$ s.t. the aver.

$$\frac{1}{N}\sum_{n=1}^{N} a_n T^n f$$

conv. $\forall$ erg. (X,μ,T) $\forall f \in L^1$.

Mean conv.: (a_n) good $\overset{\text{exerc.}}{\Longleftrightarrow}$ $\frac{1}{N}\sum_{n=1}^{N} a_n \lambda^n$ conv. $\forall \lambda \in \mathbb{T}$

exerc. (for $f \in L^2$, T invert) – use spectral thm.



Pointw. conv.: no charact.

$\boxed{\text{Ex}}$ (λ^n) good $\forall \lambda \in \mathbb{T}$:

$Y := X \times \mathbb{T}$, $S^n(x,z) = (T^n x, \lambda^n z)$, $g(x,z) := f(x)\cdot z$

Then $\frac{1}{N}\sum_{n=1}^{N} \underbrace{(S^n g)(x,z)}_{g(T^n x,\, \lambda^n z)} = \underbrace{\frac{1}{N}\sum_{n=1}^{N} \lambda^n f(T^n x)}\, z$ conv. for a.e. x and z

More:

$\boxed{\text{Thm}}$ (Wiener–Wintner '41) $\forall$ erg (X,μ,T) $\forall f \in L^1$ $\exists X' \subset X$, $\mu(X') = 1$

$$\frac{1}{N}\sum_{n=1}^{N} \lambda^n f(T^n x).$$

conv. $\forall x \in X'$ $\forall \lambda \in \mathbb{T}$.

<u>Proof (sketch)</u>: Recall:

$$L^2 = \overline{\lim}\{EF \text{ of } T\} \overset{\perp}{\oplus} \{\text{w. mixing fcts}\}$$

f: $\frac{1}{N}\sum_{n=1}^{N} |\langle T^n f, f\rangle| \longrightarrow 0$

- f EF: $Tf = \nu f$ + appr. arg. $\qquad \frac{1}{N}\sum_{n=1}^{N} \underbrace{\lambda^n \nu^n}_{(\lambda\nu)^n} f$ conv.

$\frac{1}{N}\sum_{n=1}^{N} \lambda^n \longrightarrow \begin{cases} 1, & \lambda = 1 \\ 0 \end{cases}$

- f w. mixing: Van der Corput

Let $x \in X$, $u_n := \lambda^n (T^n f)(x) \in \mathbb{C}$ bdd if $f \in L^\infty$

$$\left|\frac{1}{N}\sum_{n=1}^{N} u_{n+k}\overline{u_n}\right| = \left|\frac{1}{N}\sum_{n=1}^{N} \cancel{\lambda^{n+k}} (T^{n+k} f)(x) \cdot \cancel{\overline{\lambda}^n} (T^n \overline{f})(x)\right|$$

$$= \left|\frac{1}{N}\sum_{n=1}^{N} \cancel{\lambda^k} (T^n(T^k f \cdot \overline{f}))(x)\right| \xrightarrow[\text{Birkhoff}]{} \left|\int T^k f \cdot \overline{f}\right| = \gamma_k$$

(a.e. x indep. of λ)

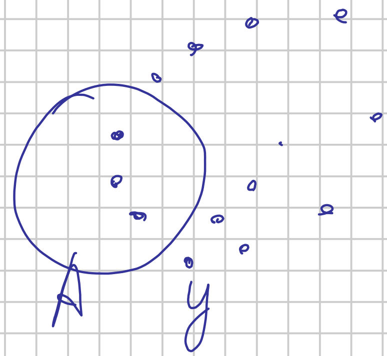
$$\frac{1}{N}\sum_{k=1}^{N} \gamma_k = \frac{1}{N}\sum_{k=1}^{N} |\langle T^k f, f\rangle| \xrightarrow[f \text{ w. mixing}]{} 0$$

Van der Corput: $\frac{1}{N}\sum_{n=1}^{N} u_n \longrightarrow 0$.
+ Appr. arg. $\blacksquare$

Rem. proof $\Rightarrow$ unif. conv. to 0 (in x) for w. mixing fcts for uniq. erg. systems (ex exc.)

$\boxed{\text{More ex.}}$ (pointwise)

- $\left(\lambda^{p(n)}\right)$, $p \in \mathbb{Z}[\cdot]$ – Lesigne '93 (WW family)
- nilsequences (seq. coming from rotations on nilpotent groups)
  $\forall g \in C(X)$ $\forall y \in Y$ $(g(S^n y))$ good, even WW family (Host–Kra '09)
- return time seq. $\forall$ erg. (Y, ν, S), $g \in L^1$, then
  $$a_n = g(S^n y)$$
  is good for a.e. y (Bourgain '89)
  $\left(g = \mathbb{1}_A, \quad \mathbb{1}_A(S^n y) = \begin{cases} 1, & S^n y \in A \\ 0 \end{cases} \right)$



- certain (many) log-exp fcts (T. E. – Draus 1/4)

  $\left(e^{2\pi i n^{\alpha}}\right)$ good , $\left(e^{2\pi i \log n}\right)$ bad

- Modified von Mangoldt fct :

$$\Lambda'(n) = \log n \cdot \mathbb{1}_{\mathbb{P}}$$

weighted aver. $\longleftrightarrow$ aver. along primes:

$\frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) T^n f$ $\qquad$ $\frac{1}{N}\sum_{n=1}^{N} T^{p_n} f$ – conv. a.e. in $L^p, p>1$ Bourgain '88, Wierdl '88

(everywhere conv. for nilsystems)

- Möbius fct

$$\mu(n) := \begin{cases} 1, & n = p_1 \cdot \ldots \cdot p_{2k}, \ p_j \text{ distinct} \\ -1, & n = p_1 \cdot \ldots \cdot p_{2k+1}, \ \text{—"—} \\ 0, & n \text{ not squarefree} \end{cases}$$

1	2	3	4	5	6	7	8	9	...
1	-1	-1	0	-1	1	-1	0	0	

$(\mu(n))$ is good with limit 0 : Sarnak '11
El Abdalaoui, Kulaga, Lemanczyk, de la Rue '14

$\boxed{\text{Conjecture}}$ (Sarnak 2011)

$\forall$ TDS (X,T) with top. entropy 0 ("good") $\forall f \in C(X)$

$$\frac{1}{N}\sum_{n=1}^{N} \mu(n)(T^n f)(x) \longrightarrow 0$$

$\forall x \in X$. Or:

$\mu \perp$ deterministic seq.

Sarnak's conj.:

1) holds for many systems: rotations, nilsystems, uniq. erg. systems, ...

2) has connection to Chowla's conj.:

$$\frac{1}{N}\sum_{n=1}^{N} \mu(n+a_1)\cdot\ldots\cdot\mu(n+a_k) \longrightarrow 0$$

$\forall k$ $\forall a_1 < \ldots < a_k$

- Chowla $\Rightarrow$ Sarnak (Sarnak '11)

- Sarnak $\Rightarrow$ Chowla for some (N_j) (Gomilko, Kwietniak, Lemańczyk '17)
  large —"— (Tao)
- Logarithmic versions are equiv.

$$\left(\frac{1}{N} \sum_{n=1}^{N} a_n \rightsquigarrow \frac{\sum_{n=1}^{N} \frac{a_n}{n}}{\sum_{n=1}^{N} \frac{1}{n}} \right) \quad \sim \log N$$

- Log. Chowla holds for $k=2$ (Tao '16)
  and odd k (Tao, Teräväinen '17)